

# ALGORITHMS FOR COMPUTING MULTIPLIER IDEALS

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ABSTRACT. We give algorithms for computing multiplier ideals using Gröbner bases in Weyl algebras. The algorithms are based on a newly introduced notion which is a variant of Budur–Mustață–Saito’s (generalized) Bernstein–Sato polynomial. We present several examples computed by our algorithms.

## CONTENTS

1. Introduction	1
2. Preliminaries	2
2.1. Gröbner bases in Weyl algebras	2
2.2. Bernstein-Sato polynomials	5
2.3. V-filtrations	6
2.4. Multiplier ideals	7
3. Algorithms for computing generalized Bernstein-Sato polynomials	8
4. Algorithms for computing multiplier ideals	13
5. Examples	16
References	22

## 1. INTRODUCTION

Multiplier ideals are an important tool in higher-dimensional algebraic geometry, and one can view them as measuring singularities. They are defined as follows: let  $X$  be a smooth complex variety and  $\mathfrak{a} \subseteq \mathcal{O}_X$  be an ideal sheaf of  $X$ . Suppose that  $\pi : \tilde{X} \rightarrow X$  is a log resolution of  $\mathfrak{a}$ , that is,  $\pi$  is a proper birational morphism,  $\tilde{X}$  is smooth and  $\pi^{-1}V(\mathfrak{a}) = F$  is a divisor with simple normal crossing support. If  $K_{\tilde{X}/X}$  is the relative canonical divisor of  $\pi$ , then the multiplier ideal of  $\mathfrak{a}$  with exponent  $c \in \mathbb{R}_{\geq 0}$  is

$$\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(c \cdot \mathfrak{a}) = \pi_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor cF \rfloor) \subseteq \mathcal{O}_X.$$

A positive rational number  $c$  is called a *jumping coefficient* if  $\mathcal{J}(\mathfrak{a}^{c-\varepsilon}) \neq \mathcal{J}(\mathfrak{a}^c)$  for all  $\varepsilon > 0$ , and the minimal jumping coefficient is called the *log-canonical threshold* of  $\mathfrak{a}$  and denoted by  $\text{lct}(\mathfrak{a})$ . Since multiplier ideals are defined via log resolutions, it is difficult to compute them in general (when the ideal  $\mathfrak{a}$  is a monomial ideal or a principal ideal generated by a non-degenerate polynomial, there is a combinatorial description of the multiplier ideals  $\mathcal{J}(\mathfrak{a}^c)$ . See [4] and [5]). In this paper, we will give algorithms for computing multiplier ideals using the theory of  $D$ -modules.

The Bernstein-Sato polynomial (or  $b$ -function) is one of the main objects in the theory of  $D$ -modules. It has turned out that jumping coefficients are deeply related to Bernstein-Sato polynomials. For a given polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ , the Bernstein-Sato polynomial  $b_f(s)$  of  $f$  is the monic polynomial in one variable  $b(s) \in \mathbb{C}[s]$  of minimal degree having the property that there exists a linear differential operator  $P(x, s)$  such that  $b(s)f^s = P(x, s)f^{s+1}$ . Kollár [7] proved that the log canonical threshold of  $f$  is the minimal root of  $b_f(-s)$ . Furthermore, Ein-Lazarsfeld-Smith-Varolin [3] extended Kollár's result to higher jumping coefficients: they proved that all jumping coefficients in the interval  $(0, 1]$  are roots of  $b_f(-s)$ . Recently Budur-Mustața-Saito introduced the notion of Bernstein-Sato polynomials of arbitrary ideal sheaves using the theory of  $V$ -filtrations of Kashiwara [6] and Malgrange [9]. They then gave a criterion for membership of multiplier ideals in terms of their Bernstein-Sato polynomials, and proved that all jumping coefficients of  $\mathfrak{a} \subset \mathcal{O}_X$  in the interval  $(\text{lct}(\mathfrak{a}), \text{lct}(\mathfrak{a}) + 1]$  are roots of the Bernstein-Sato polynomial of  $\mathfrak{a}$  up to sign.

It is difficult to compute Bernstein-Sato polynomials in general, but Oaku [11], [12], [13] gave algorithms for computing Bernstein-Sato polynomials  $b_f(s)$  using Gröbner bases in Weyl algebras (algorithms for computing Gröbner bases in Weyl algebras are implemented in some computer systems, such as Kan/Sm1 [20] and Risa/Asir [10]). In this paper, we give algorithms for computing Budur-Mustața-Saito's Bernstein-Sato polynomials (Theorems 3.4, 3.5 and 3.7). Our algorithms are natural generalizations of Oaku's algorithm.

The other ingredient of this paper is algorithms for computing multiplier ideals. The algorithm for computing generalized Bernstein-Sato polynomials enables us to solve the membership problem for multiplier ideals, but does not give a system of generators of multiplier ideals. We modify the definition of Budur-Mustața-Saito's Bernstein-Sato polynomials to determine a system of generators of the multiplier ideals of a given ideal (Definition 4.1). Then we obtain algorithms for computing our Bernstein-Sato polynomials and algorithms for computing multiplier ideals (Theorem 4.4, Theorem 4.5). Our algorithms are based on the theory of Gröbner bases in Weyl algebras (see [14] and [18] for a review of Gröbner bases in Weyl algebras and their applications). We conclude the paper by presenting several examples computed by our algorithms.

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## 2. PRELIMINARIES

**2.1. Gröbner bases in Weyl algebras.** We denote by  $\mathbb{C}$  the complex number field. When we use a computer algebra system, we may work with a computable field  $\mathbb{Q}(z_1, \dots, z_l) \subset \mathbb{C}$  which is sufficient to express objects that appear in the computations. Let  $X$  be the affine space  $\mathbb{C}^d$  with the coordinate system  $\mathbf{x} = (x_1, \dots, x_d)$ ,

and  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_d]$  a polynomial ring over  $\mathbb{C}$  which is the coordinate ring of  $X$ . We denote by  $\partial_{\mathbf{x}} = (\partial_{x_1}, \dots, \partial_{x_d})$  the partial differential operators where  $\partial_{x_i} = \frac{\partial}{\partial x_i}$ . We set

$$D_X = \mathbb{C}\langle \mathbf{x}, \partial_{\mathbf{x}} \rangle = \mathbb{C}\langle x_1, \dots, x_d, \partial_{x_1}, \dots, \partial_{x_d} \rangle,$$

the rings of differential operators of  $X$ , and call it the *Weyl algebra* (in  $d$  variables). This ring is non-commutative  $\mathbb{C}$ -algebra with the commutation rules

$$x_i x_j = x_j x_i, \quad \partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}, \quad \partial_{x_i} x_j = x_j \partial_{x_i} \text{ for } i \neq j, \text{ and } \partial_{x_i} x_i = x_i \partial_{x_i} + 1.$$

We write  $\langle P_1, \dots, P_r \rangle$  for the left ideal of  $D_X$  generated by  $P_1, \dots, P_r \in D_X$ . We use the notation  $\mathbf{x}^{\boldsymbol{\mu}} = \prod_{i=1}^d x_i^{\mu_i}$ , and  $\partial_{\mathbf{x}}^{\boldsymbol{\nu}} = \prod_{i=1}^d \partial_{x_i}^{\nu_i}$  for  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$ ,  $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{Z}_{\geq 0}^d$ . We denote by  $|\boldsymbol{\mu}| := \mu_1 + \dots + \mu_d$  the side of  $\boldsymbol{\mu}$ . We call a real vector  $(\mathbf{v}, \mathbf{w}) = (v_1, \dots, v_d, w_1, \dots, w_d) \in \mathbb{R}^d \times \mathbb{R}^d$  a *weight vector* if

$$v_i + w_i \geq 0 \text{ for } i = 1, 2, \dots, d.$$

We define the ascending filtration  $\dots \subset F_0^{(\mathbf{v}, \mathbf{w})} D_X \subset F_1^{(\mathbf{v}, \mathbf{w})} D_X \subset \dots$  on  $D_X$  with respect to the weight vector  $(\mathbf{v}, \mathbf{w})$  by

$$F_m^{(\mathbf{v}, \mathbf{w})} D_X = \left\{ \sum_{\mathbf{v} \cdot \boldsymbol{\mu} + \mathbf{w} \cdot \boldsymbol{\nu} \leq m} a_{\boldsymbol{\mu}\boldsymbol{\nu}} \mathbf{x}^{\boldsymbol{\mu}} \partial_{\mathbf{x}}^{\boldsymbol{\nu}} \mid a_{\boldsymbol{\mu}\boldsymbol{\nu}} \in \mathbb{C} \right\} \subset D_X$$

where  $\mathbf{v} \cdot \boldsymbol{\mu} = \sum v_i \mu_i$  is the usual inner product of  $\mathbf{v}$  and  $\boldsymbol{\mu}$ . Then we have

$$F_{m_1}^{(\mathbf{v}, \mathbf{w})} D_X \cdot F_{m_2}^{(\mathbf{v}, \mathbf{w})} D_X \subset F_{m_1+m_2}^{(\mathbf{v}, \mathbf{w})} D_X$$

for all  $m_1, m_2 \in \mathbb{Z}$  by the conditions  $v_i + w_i \geq 0$  and the commutation rules of  $D_X$ . In particular,  $F_0^{(\mathbf{v}, \mathbf{w})} D_X$  is a sub-ring of  $D_X$ , and  $F_m^{(\mathbf{v}, \mathbf{w})} D_X$ 's are  $F_0^{(\mathbf{v}, \mathbf{w})} D_X$ -submodules of  $D_X$ . We can define the associated graded ring of  $D_X$  with respect to the filtration

$$\text{Gr}^{(\mathbf{v}, \mathbf{w})} D_X = \bigoplus_{m \in \mathbb{Z}} F_m^{(\mathbf{v}, \mathbf{w})} D_X / F_{m-1}^{(\mathbf{v}, \mathbf{w})} D_X.$$

**Definition 2.1.** The *order* of  $P \in D_X$  is defined by

$$\text{ord}_{(\mathbf{v}, \mathbf{w})}(P) = \min\{m \mid P \in F_m^{(\mathbf{v}, \mathbf{w})} D_X\}.$$

For a non-zero  $P \in D_X$  with  $\text{ord}_{(\mathbf{v}, \mathbf{w})}(P) = m$ , the *initial form*  $\text{in}_{(\mathbf{v}, \mathbf{w})}(P)$  of  $P$  is the image of  $P$  in  $\text{Gr}_m^{(\mathbf{v}, \mathbf{w})} D_X := F_m^{(\mathbf{v}, \mathbf{w})} D_X / F_{m-1}^{(\mathbf{v}, \mathbf{w})} D_X$ . For a left ideal  $I \subset D_X$ , the *initial ideal*  $\text{in}_{(\mathbf{v}, \mathbf{w})}(I)$  of  $I$  is the left ideal of  $\text{Gr}^{(\mathbf{v}, \mathbf{w})} D_X$  generated by all initial forms of elements in  $I$ . A finite subset  $G$  of  $D_X$  is called *Gröbner basis* of  $I$  with respect to  $(\mathbf{v}, \mathbf{w})$  if  $I$  is generated by  $G$  and  $\text{in}_{(\mathbf{v}, \mathbf{w})}(I)$  is generated by initial forms of elements in  $G$ .

It is known that there is an algorithm for computing Gröbner bases ([18] Algorithm 1.2.6). We can compute the restriction of ideals to sub-algebras using Gröbner bases as in the commutative case.

**Lemma 2.2.** *Let  $Z$  be a subsystem of  $(\mathbf{x}, \partial_{\mathbf{x}})$ , and  $\mathbb{C}\langle Z \rangle$  a sub-algebra of  $D_X$  generated by  $Z$  over  $\mathbb{C}$ . Let  $(\mathbf{v}, \mathbf{w})$  be a weight vector such that  $v_i > 0$  (resp.  $w_j > 0$ ) if  $x_i$  (resp.  $\partial_{x_j}$ ) is not a member of  $Z$ , and  $v_i = 0$  (resp.  $w_j = 0$ ) otherwise. Let  $I$  be*

a left ideal of  $D_X$  and  $G$  a Gröbner basis of with respect to  $(\mathbf{v}, \mathbf{w})$ , then  $G \cap \mathbb{C}\langle Z \rangle$  is a system of generators of the left ideal  $I \cap \mathbb{C}\langle Z \rangle$ .

We can also compute the intersection of ideals using elimination of variables as in the commutative case.

**Lemma 2.3.** *Let  $I$  and  $J$  be left ideals of  $D_X$ . Then*

$$I \cap J = D_X[u](uI + (1 - u)J) \cap D_X.$$

*Proof.* If  $P \in I \cap J$ , then  $P = uP + (1 - u)P \in D_X[u](uI + (1 - u)J) \cap D_X$ . Let  $P \in D_X[u](uI + (1 - u)J) \cap D_X$ . By substituting 1 and 0 to  $u$ , we have  $P \in I$  and  $P \in J$ .  $\square$

Note that substituting some  $p \in D_X$  to the variable  $u$  makes sense only when  $p$  is in the center of  $D_X$ , that is,  $p \in \mathbb{C}$ . In this case, the left ideal of  $D_X[u]$  generated by  $u - p$  is a two-side ideal.

From now on, we assume that the weight vector  $(\mathbf{v}, \mathbf{w})$  satisfies

$$v_i + w_i = 0 \text{ for } i = 1, 2, \dots, d.$$

Then  $D_X$  has a structure of a graded algebra: We set

$$[D_X]_m^{(\mathbf{v}, \mathbf{w})} := \left\{ \sum_{\mathbf{v} \cdot \boldsymbol{\mu} + \mathbf{w} \cdot \boldsymbol{\nu} = m} a_{\boldsymbol{\mu}\boldsymbol{\nu}} \mathbf{x}^{\boldsymbol{\mu}} \partial_{\mathbf{x}}^{\boldsymbol{\nu}} \mid a_{\boldsymbol{\mu}\boldsymbol{\nu}} \in \mathbb{C} \right\} \subset D_X.$$

Then  $F_m^{(\mathbf{v}, \mathbf{w})} D_X = \bigoplus_{k \leq m} [D_X]_k^{(\mathbf{v}, \mathbf{w})}$  and  $\text{Gr}_m^{(\mathbf{v}, \mathbf{w})} D_X \cong [D_X]_m^{(\mathbf{v}, \mathbf{w})}$  since the commutation rules of  $D_X$  are homogeneous of weight 0. Hence  $D_X$  is a graded algebra  $D_X = \bigoplus_{m \in \mathbb{Z}} [D_X]_m^{(\mathbf{v}, \mathbf{w})}$  and isomorphic to  $\text{Gr}^{(\mathbf{v}, \mathbf{w})} D_X$ . In particular  $[D_X]_0^{(\mathbf{v}, \mathbf{w})}$  is a sub-ring of  $D_X$ . We call an element in  $[D_X]_m^{(\mathbf{v}, \mathbf{w})}$  a *homogeneous element* of degree  $m$ . A left ideal  $J$  of  $D_X$  is called a *homogeneous ideal* if  $J$  is generated by homogeneous elements.

**Definition 2.4.** For  $P = \sum P_m \in D_X$  with  $P_m \in [D_X]_m^{(\mathbf{v}, \mathbf{w})}$  and  $m_0 = \text{ord}_{(\mathbf{v}, \mathbf{w})}(P)$ , we define the homogenization of  $P$  with homogenizing variable  $u_1$  to be

$$P^h = \sum P_m u_1^{m_0 - m} \in D_X[u_1].$$

where For a left ideal  $J$  of  $D_X$ , we define the homogenization of  $J$  to be the left ideal of  $D_X[u_1]$

$$J^h = \langle P^h \mid P \in J \rangle.$$

**Definition 2.5.** For a left ideal  $J$  of  $D_X$ , we set

$$J^* = J^h \cap D_X = \bigoplus_{m \in \mathbb{Z}} (J \cap [D_X]_m^{(\mathbf{v}, \mathbf{w})}),$$

the ideal of  $D_X$  generated by all homogeneous elements in  $J$ .

**Lemma 2.6.** *Let  $J = \langle P_1, \dots, P_r \rangle$  be a left ideal of  $D_X$ . Then*

$$J^* = D_X[u_1, u_2] \langle P_1^h, \dots, P_r^h, u_1 u_2 - 1 \rangle \cap D_X.$$

*Proof.* It is easy to see that

$$J^h = (D_X[u_1, u_1^{-1}]\langle P_1^h, \dots, P_r^h \rangle) \cap D_X[u_1] = \langle P_1^h, \dots, P_r^h, u_1 u_2 - 1 \rangle \cap D_X[u_1].$$

Since  $J^* = J^h \cap D_X$ , we obtain the assertion.  $\square$

**2.2. Bernstein-Sato polynomials.** Budur–Mustaǎ–Saito introduced generalized Bernstein-Sato polynomials (or  $b$ -function) of arbitrary varieties in [2] and proved relations between generalized Bernstein-Sato polynomials and multiplier ideals using the theory of the  $V$ -filtration of Kashiwara and Malgrange.

Let  $X$  be the affine space  $\mathbb{C}^n$  with the coordinate ring  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$ , and fix an ideal  $\mathfrak{a}$  of  $\mathbb{C}[\mathbf{x}]$  with a system of generators  $\mathbf{f} = (f_1, \dots, f_r)$ . Let  $Y = X \times \mathbb{C}^r$  be the affine space  $\mathbb{C}^{n+r}$  with the coordinate system  $(\mathbf{x}, \mathbf{t}) = (x_1, \dots, x_n, t_1, \dots, t_r)$ . Then  $X \times \{0\} = V(t_1, \dots, t_r) \cong X$  is a linear subspace of  $Y$  with the defining ideal  $I_{X \times \{0\}} = \langle t_1, \dots, t_r \rangle$ . We denote the rings of differential operators of  $X$  and  $Y$  by

$$D_X = \mathbb{C}\langle \mathbf{x}, \partial_{\mathbf{x}} \rangle = \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle,$$

$$D_Y = \mathbb{C}\langle \mathbf{x}, \mathbf{t}, \partial_{\mathbf{x}}, \partial_{\mathbf{t}} \rangle = \mathbb{C}\langle x_1, \dots, x_n, t_1, \dots, t_r, \partial_{x_1}, \dots, \partial_{x_n}, \partial_{t_1}, \dots, \partial_{t_r} \rangle.$$

We use the notation  $\mathbf{x}^{\mu_1} = \prod_{i=1}^n x_i^{\mu_{1i}}$ ,  $\mathbf{t}^{\mu_2} = \prod_{j=1}^r t_j^{\mu_{2j}}$ ,  $\partial_{\mathbf{x}}^{\nu_1} = \prod_{i=1}^n \partial_{x_i}^{\nu_{1i}}$ , and  $\partial_{\mathbf{t}}^{\nu_2} = \prod_{j=1}^r \partial_{t_j}^{\nu_{2j}}$  for  $\mu_1 = (\mu_{11}, \dots, \mu_{1n})$ ,  $\nu_1 = (\nu_{11}, \dots, \nu_{1n}) \in \mathbb{Z}_{\geq 0}^n$  and  $\mu_2 = (\mu_{21}, \dots, \mu_{2r})$ ,  $\nu_2 = (\nu_{21}, \dots, \nu_{2r}) \in \mathbb{Z}_{\geq 0}^r$ . The  $\mathbb{C}[\mathbf{x}]$ -module  $N_{\mathbf{f}} := \mathbb{C}[\mathbf{x}][\prod_i f_i^{-1}, s_1, \dots, s_r] \prod_i f_i^{s_i}$ , where  $s_i$ 's are independent variables and  $\prod_i f_i^{s_i}$  is a symbol, has a  $D_X$ -module structure as follows: The action of  $\mathbb{C}[\mathbf{x}]$  on  $N_{\mathbf{f}}$  is given by the canonical one, and the action of  $\partial_{x_i}$  are given by

$$\partial_{x_j}(h \prod_i f_i^{s_i}) = \left( \partial_{x_j}(h) + h \sum_{k=1}^r s_j \frac{\partial_{x_j}(f_k)}{f_k} \right) \prod_i f_i^{s_i}.$$

for  $h \in \mathbb{C}[\mathbf{x}][\prod_i f_i^{-1}, s_1, \dots, s_r]$ . This action is defined formally, but it has an obvious meaning when some integers are substituted for  $s_i$ 's. We define  $D_X$ -linear actions  $t_j$  and  $\partial_{t_j}$  on  $N_{\mathbf{f}}$  by

$$t_j(h(x, s_1, \dots, s_r) \prod_i f_i^{s_i}) = h(x, s_1, \dots, s_j + 1, \dots, s_r) f_j \prod_i f_i^{s_i}$$

and

$$\partial_{t_j}(h(x, s_1, \dots, s_r) \prod_i f_i^{s_i}) = -s_j h(x, s_1, \dots, s_j - 1, \dots, s_r) f_j^{-1} \prod_i f_i^{s_i}.$$

for  $h(x, s_1, \dots, s_r) \in \mathbb{C}[\mathbf{x}][\prod_i f_i^{-1}, s_1, \dots, s_r]$ . Then it follows that  $N_{\mathbf{f}}$  is a  $D_Y$ -module because the actions defined above respect the commutation rules of  $D_Y$ . Note that  $-\partial_{t_i} t_i \prod_i f_i^{s_i} = s_i \prod_i f_i^{s_i}$  for all  $i$ .

**Definition 2.7** ([2]). Let  $\sigma = -(\sum_i \partial_{t_i} t_i)$ , and let  $s$  be a new variable. Then the global generalized Bernstein-Sato polynomial  $b_{\mathfrak{a},g}(s) \in \mathbb{C}[s]$  of  $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$  and  $g \in \mathbb{C}[\mathbf{x}]$  is defined to be the monic polynomial of minimal degree satisfying

$$b(\sigma)g \prod_i f_i^{s_i} = \sum_{j=1}^r P_j g f_j \prod_i f_i^{s_i} \quad (2.1)$$

for some  $P_j \in D_X \langle -\partial_{t_j} t_k \mid 1 \leq j, k \leq r \rangle$ . We define  $b_{\mathfrak{a}}(s) = b_{\mathfrak{a},1}(s)$ .

For a prime ideal  $\mathfrak{p}$  of  $\mathbb{C}[\mathbf{x}]$ , we define the local generalized Bernstein-Sato polynomial  $b_{\mathfrak{a},g}^{\mathfrak{p}}(s)$  at  $\mathfrak{p}$  to be the monic polynomial of minimal degree satisfying

$$b(\sigma)gh \prod_i f_i^{s_i} = \sum_{j=1}^r P_j g f_j \prod_i f_i^{s_i}$$

for some  $P_j \in D_X \langle -\partial_{t_j} t_k \mid 1 \leq j, k \leq r \rangle$  and  $h \notin \mathfrak{p}$ . We define  $b_{\mathfrak{a}}^{\mathfrak{p}}(s) = b_{\mathfrak{a},1}^{\mathfrak{p}}(s)$ .

Note that  $\mathbb{C}[\mathbf{x}]_{\mathfrak{p}} \otimes_{\mathbb{C}[\mathbf{x}]} D_X$  is the ring of differential operators of  $\text{Spec } \mathbb{C}[\mathbf{x}]_{\mathfrak{p}}$ . It is proved in [2] that generalized Bernstein-Sato polynomials are well-defined, that is, they do not depend on the choice of generators of  $\mathfrak{a}$ , and all their roots are negative rational numbers. These facts follow from the theory of  $V$ -filtrations of Kashiwara [6] and Malgrange [9]. When  $\mathfrak{a}$  is a principal ideal generated by  $f$ , then  $b_{\mathfrak{a}}(s)$  coincides with the classical Bernstein-Sato polynomial  $b_f(s)$  of  $f$ .

**2.3. V-filtrations.** We will briefly recall the definition and some basic properties of  $V$ -filtrations. See [9], [6], [15] and [2] for details.

We fix the weight vector  $(\mathbf{w}, -\mathbf{w}) \in \mathbb{Z}^{2(n+r)}$ ,  $\mathbf{w} = ((0, \dots, 0), (1, \dots, 1)) \in \mathbb{Z}^n \times \mathbb{Z}^r$ , that is, we assign the weight 1 to  $\partial_{t_j}$ ,  $-1$  to  $t_j$ , and 0 to  $x_i$  and  $\partial_{x_i}$ . Then

$$F_m^{(\mathbf{w}, -\mathbf{w})} D_Y = \left\{ \sum_{-|\boldsymbol{\mu}_2| + |\boldsymbol{\nu}_2| \leq m} a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \mathbf{x}^{\boldsymbol{\mu}_1} \mathbf{t}^{\boldsymbol{\mu}_2} \partial_{\mathbf{x}}^{\boldsymbol{\nu}_1} \partial_{\mathbf{t}}^{\boldsymbol{\nu}_2} \mid a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \in \mathbb{C} \right\}$$

In this paper, we call the decreasing filtration  $V^m D_Y := F_{-m}^{(\mathbf{w}, -\mathbf{w})} D_Y$  on  $D_Y$  the  $V$ -filtration of  $D_Y$  along  $X \times \{0\}$  (some author call the increasing filtration  $F^{(\mathbf{w}, -\mathbf{w})}$  the  $V$ -filtration). Note that

$$\begin{aligned} V^m D_Y &= \left\{ \sum_{|\boldsymbol{\mu}_2| - |\boldsymbol{\nu}_2| \geq m} a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \mathbf{x}^{\boldsymbol{\mu}_1} \mathbf{t}^{\boldsymbol{\mu}_2} \partial_{\mathbf{x}}^{\boldsymbol{\nu}_1} \partial_{\mathbf{t}}^{\boldsymbol{\nu}_2} \mid a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \in \mathbb{C} \right\} \\ &= \{ P \in D_Y \mid P(I_{X \times \{0\}})^j \subset (I_{X \times \{0\}})^{j+m} \text{ for any } j \geq 0 \}, \end{aligned}$$

with the convention  $I_{X \times \{0\}}^j = \mathbb{C}[\mathbf{x}, \mathbf{t}]$  for all  $j \leq 0$ .

**Definition 2.8.** The  $V$ -filtration along  $X \times \{0\}$  on a finitely generated left  $D_Y$ -module  $M$  is an exhaustive decreasing filtration  $\{V^{\alpha} M\}_{\alpha \in \mathbb{Q}}$  indexed by  $\mathbb{Q}$ , such that:

- (i)  $V^{\alpha} M$  are finitely generated  $V^0 D_Y$ -submodules of  $M$ .
- (ii)  $\{V^{\alpha} M\}_{\alpha}$  is left-continuous and discrete, that is,  $V^{\alpha} M = \bigcap_{\alpha' < \alpha} V^{\alpha'} M$ , and every interval contains only finitely many  $\alpha \in \mathbb{Q}$  with  $\text{Gr}_V^{\alpha} M \neq 0$ . Here  $\text{Gr}_V^{\alpha} M := V^{\alpha} M / (\bigcup_{\alpha' > \alpha} V^{\alpha'} M)$ .
- (iii)  $(V^i D_Y)(V^{\alpha} M) \subset V^{\alpha+i} M$  for any  $i \in \mathbb{Z}$ ,  $\alpha \in \mathbb{Q}$ .
- (iv)  $(V^i D_Y)(V^{\alpha} M) = V^{\alpha+i} M$  for any  $i > 0$  if  $\alpha \gg 0$ .
- (v) the action of  $\sigma + \alpha$  is nilpotent on  $\text{Gr}_V^{\alpha} M$ .

**Remark 2.9.** (i) The filtration  $V$  is unique if it exists ([6]), and  $D_Y$ -submodule  $D_Y \prod_i f_i^{s_i} \cong D_Y / \text{Ann}_{D_Y} \prod_i f_i^{s_i}$  of  $N_{\mathbf{f}}$  has such a  $V$ -filtration (see [2]).

(ii) For  $\alpha \neq z \in \mathbb{C}$ , the action of  $\sigma + z$  on  $\text{Gr}_V^{\alpha} M$  is invertible. Hence, if  $\text{Gr}_V^{\alpha} M \neq 0$ ,

$u \notin V^{\alpha+\varepsilon}M$  and  $b(\sigma)u \in V^{\alpha+\varepsilon}M$  for some  $b(s) \in \mathbb{C}[s]$  and all sufficiently small  $\varepsilon > 0$ , then  $s + \alpha$  is a factor of  $b(s)$ .

Let  $\iota : X \rightarrow Y$  be the graph embedding  $x \mapsto (x, f_1(x), \dots, f_r(x))$  of  $\mathbf{f} = (f_1, \dots, f_r)$ , and  $M_{\mathbf{f}} = \iota_+ \mathbb{C}[\mathbf{x}]$ , where  $\iota_+$  denotes the direct image for left  $D$ -modules. There is a natural isomorphism  $M_{\mathbf{f}} \cong \mathbb{C}[\mathbf{x}] \otimes_{\mathbb{C}} \mathbb{C}[\partial_{t_1}, \dots, \partial_{t_r}]$  (see [1]), and the action of  $\mathbb{C}[\mathbf{x}]$  and  $\partial_{t_1}, \dots, \partial_{t_r}$  on  $M_{\mathbf{f}}$  is given by the canonical one, and the action of a vector field  $\xi$  on  $X$  and  $t_j$  are given by

$$\begin{aligned} \xi(g \otimes \partial_t^\nu) &= \xi g \otimes \partial_t^\nu - \sum_j (\xi f_j) g \otimes \partial_{t_j} \partial_t^\nu, \\ t_j(g \otimes \partial_t^\nu) &= f_j g \otimes \partial_t^\nu - \nu_j g \otimes \partial_t^{\nu-1_j}. \end{aligned}$$

where  $1_j$  is the element of  $\mathbb{Z}^r$  whose  $i$ -th component is 1 if  $i = j$  and 0 otherwise.

**Definition 2.10** ([2]). Let  $M$  be a  $D_Y$ -module with  $V$ -filtration. For  $u \in M$ , the Bernstein-Sato polynomial  $b_u(s)$  of  $u$  is the monic minimal polynomial of the action of  $\sigma$  on  $V^0 D_Y u / V^1 D_Y u$ .

By the properties of  $V$ -filtration in Definition 2.8, the induced filtration  $V$  on  $(V^0 D_Y u) / (V^1 D_Y u)$  is finite (see [2] Section 2.1). This guarantees the existence of  $b_u(s)$ . If  $u \in V^\alpha M$ , then  $V^0 D_Y u \subset V^\alpha M$  and  $V^1 D_Y u \subset V^{\alpha+1} M$ . Hence, if we set  $\alpha_0 = \max\{\alpha \mid u \in V^\alpha M\}$ , then  $u \notin V^{\alpha_0+\varepsilon} M$  and  $b_u(\sigma)u \in V^1 D_Y u \subset V^{\alpha_0+1} M \subset V^{\alpha_0+\varepsilon} M$  for sufficiently small  $\varepsilon > 0$ . Hence

$$\max\{\alpha \mid u \in V^\alpha M\} = \min\{\alpha \mid \text{Gr}_V^\alpha((V^0 D_Y u)) \neq 0\} \min\{\alpha \mid b_u(-\alpha) = 0\}.$$

Therefore we conclude the next proposition.

**Proposition 2.11** ([15]). *Let  $M$  be a  $D_Y$ -module with  $V$ -filtration. Then*

$$V^\alpha M = \{u \in M \mid \alpha \leq \alpha' \text{ if } b_u(-\alpha') = 0\}.$$

Since we have a canonical injection  $M_{\mathbf{f}} \rightarrow N_{\mathbf{f}} = \mathbb{C}[\mathbf{x}] [\prod_i f_i^{-1}, s_1, \dots, s_r] \prod_i f_i^{s_i}$  that sends  $g \otimes \partial_t^\nu$  to  $g \partial_t^\nu \prod_i f_i^{s_i}$ , the generalized Bernstein-Sato polynomial  $b_{a,g}(s)$  coincides with  $b_u(s)$  where  $u = g \otimes 1 \in M_{\mathbf{f}}$  (see Observation 3.1 in the next section).

**2.4. Multiplier ideals.** We will recall the relations between generalized Bernstein-Sato polynomials and multiplier ideals following [2]. The reader is referred to [8] for general properties of multiplier ideals. For a positive rational number  $c$ , the multiplier ideal  $\mathcal{J}(\mathbf{a}^c)$  is defined via a log resolution of  $\mathbf{a}$ . Let  $\pi : \tilde{X} \rightarrow X = \text{Spec } \mathbb{C}[\mathbf{x}]$  be a log resolution of  $\mathbf{a}$ , namely,  $\pi$  is a proper birational morphism,  $\tilde{X}$  is smooth, and there exists an effective divisor  $F$  on  $\tilde{X}$  such that  $\mathbf{a} \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$  and the union of the support of  $F$  and the exceptional divisor of  $\pi$  has simple normal crossings. For a given real number  $c \geq 0$ , the multiplier ideal  $\mathcal{J}(\mathbf{a}^c)$  associated to  $c$  is defined to be the ideal

$$\mathcal{J}(\mathbf{a}^c) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - \lfloor cF \rfloor))$$

where  $K_{\tilde{X}/X} = K_{\tilde{X}} - \pi^* K_X$  is the relative canonical divisor of  $\pi$ . This definition is independent of the choice of a log resolution  $\pi : \tilde{X} \rightarrow X$ . The reader is referred

to [8] for general properties of multiplier ideals. By the definition, if  $c < c'$ , then  $\mathcal{J}(\mathfrak{a}^c) \supset \mathcal{J}(\mathfrak{a}^{c'})$  and  $\mathcal{J}(\mathfrak{a}^c)$  is right-continuous in  $c$ , that is,  $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\mathfrak{a}^{c+\varepsilon})$  for sufficiently small  $\varepsilon > 0$ . The multiplier ideals give a decreasing filtration on  $\mathcal{O}_X$ , and there are rational numbers  $0 = c_0 < c_1 < c_2 < \cdots$  such that  $\mathcal{J}(\mathfrak{a}^{c_j}) = \mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c_{j+1}})$  for  $c_j \leq c < c_{j+1}$ . These  $c_j$  for  $j > 0$  are called the *jumping coefficients*, and the minimal jumping coefficient  $c_1$  is called the *log-canonical threshold* of  $\mathfrak{a}$  and denoted by  $\text{lct}(\mathfrak{a})$ . By the definition, it follows that multiplier ideals are integrally closed, and the multiplier ideal associated to the log canonical threshold is radical. It is known that  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq \lambda(\mathfrak{a})$  where  $\lambda(\mathfrak{a})$  is the analytic spread of  $\mathfrak{a}$ . Recall that analytic spread of  $\mathfrak{a}$  is the minimal number of elements needed to generate  $\mathfrak{a}$  up to integral closure, and thus  $\mu(\mathfrak{a}) \geq \lambda(\mathfrak{a})$  where  $\mu(\mathfrak{a})$  is the minimal number of generators of  $\mathfrak{a}$ . In particular, if  $\mathfrak{a}$  is a principal ideal generated by  $f$ , then  $\mathcal{J}(f^c) = f\mathcal{J}(f^{c-1})$  for  $c \geq 1$ .

Budur–Mustaǎ–Saito proved that the  $V$ -filtration on  $\mathbb{C}[x]$  is essentially equivalent to the filtration by multiplier ideals using the theory of mixed Hodge modules ([16], [17]), and gave a description of multiplier ideals in terms of generalized Bernstein-Sato polynomials.

**Theorem 2.12** ([2]). *We denote by  $V$  the filtration on  $\mathbb{C}[\mathbf{x}] \cong \mathbb{C}[\mathbf{x}] \otimes 1$  induced by the  $V$ -filtration on  $\iota_+ \mathbb{C}[\mathbf{x}]$ . Then  $\mathcal{J}(\mathfrak{a}^c) = V^{c+\varepsilon} \mathbb{C}[\mathbf{x}]$  and  $V^\alpha \mathbb{C}[\mathbf{x}] = \mathcal{J}(\mathfrak{a}^{\alpha-\varepsilon})$  for any  $\alpha \in \mathbb{Q}$  and  $0 < \varepsilon \ll 1$ . Therefore the following hold:*

(i) *For a given rational number  $c \geq 0$  and a prime ideal  $\mathfrak{p} \subset \mathbb{C}[\mathbf{x}]$ ,*

$$\begin{aligned} \mathcal{J}(\mathfrak{a}^c) &= \{g \in \mathbb{C}[\mathbf{x}] \mid c < c' \text{ if } b_{\mathfrak{a},g}(-c') = 0\}, \\ \mathcal{J}(\mathfrak{a}^c)_{\mathfrak{p}} \cap \mathbb{C}[\mathbf{x}] &= \{g \in \mathbb{C}[\mathbf{x}] \mid c < c' \text{ if } b_{\mathfrak{a},g}^{\mathfrak{p}}(-c') = 0\}. \end{aligned}$$

*In particular, the log canonical threshold  $\text{lct}(\mathfrak{a})$  of  $\mathfrak{a} = \langle f_1, \dots, f_r \rangle$  is the minimal root of  $b_{\mathfrak{a}}(-s)$ .*

(ii) *All jumping coefficients of  $\mathfrak{a}$  in  $[\text{lct}(\mathfrak{a}), \text{lct}(\mathfrak{a}) + 1)$  are roots of  $b_{\mathfrak{a}}(-s)$ .*

Therefore an algorithm for computing generalized Bernstein-Sato polynomials induces an algorithm for solving membership problem for multiplier ideals, and in particular, an algorithm for computing log canonical thresholds.

### 3. ALGORITHMS FOR COMPUTING GENERALIZED BERNSTEIN-SATO POLYNOMIALS

In this section, we obtain an algorithm for computing generalized Bernstein-Sato polynomials of arbitrary ideals. The algorithms for computing classical Bernstein-Sato polynomials are given by Oaku (see [11], [12], [13]). We will generalize Oaku's algorithm to arbitrary  $n$  and  $g$ .

Let  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$  be a polynomial ring over  $\mathbb{C}$ , and  $\mathfrak{a}$  an ideal with a system of generators  $\mathbf{f} = (f_1, \dots, f_r)$ , and we fix the weight vector  $(\mathbf{w}, -\mathbf{w}) \in \mathbb{Z}^{n+r} \times \mathbb{Z}^{n+r}$ ,  $\mathbf{w} = ((0, \dots, 0), (1, \dots, 1)) \in \mathbb{Z}^n \times \mathbb{Z}^r$ .

**Observation 3.1.** We rewrite the definition of generalized Bernstein-Sato polynomials in several ways.



Recall that  $b_{\mathbf{a},g}(s)$  is the monic polynomial of minimal degree satisfying

$$b(\sigma)g \prod_i f_i^{s_i} = \sum_{j=1}^r P_j g f_j \prod_i f_i^{s_i}$$

for some  $P_j \in D_X \langle -\partial_{t_j} t_k \mid 1 \leq j, k \leq r \rangle$  (Definition 2.7 (2.1)) Since

$$\begin{aligned} [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} &= \left\{ \sum_{|\boldsymbol{\nu}_2| - |\boldsymbol{\mu}_2| = 0} a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \mathbf{x}^{\boldsymbol{\mu}_1} \mathbf{t}^{\boldsymbol{\mu}_2} \partial_{\mathbf{x}}^{\boldsymbol{\nu}_1} \partial_{\mathbf{t}}^{\boldsymbol{\nu}_2} \mid a_{\boldsymbol{\mu}_1 \boldsymbol{\mu}_2 \boldsymbol{\nu}_1 \boldsymbol{\nu}_2} \in \mathbb{C} \right\} \\ &= D_X \langle -\partial_{t_j} t_k \mid 1 \leq j, k \leq r \rangle, \end{aligned}$$

and  $\sigma = -\sum \partial_{t_i} t_i$  is a homogeneous element of degree 0, the condition (2.1) is equivalent to saying that

$$\begin{aligned} &(b(\sigma)g - \sum_{j=1}^r P_j g f_j) \prod_i f_i^{s_i} = 0 \quad \text{for } \exists P_j \in [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} \\ \iff &b(\sigma)g - \sum_{j=1}^r P_j g f_j \in \text{Ann}_{[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}} \prod_i f_i^{s_i} \quad \text{for } \exists P_j \in [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} \\ \iff &b(\sigma)g \in \text{Ann}_{[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}} \prod_i f_i^{s_i} + [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} g \mathbf{a}. \end{aligned}$$

Since  $(I + J) \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} = I \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} + J \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$  for homogeneous ideals  $I$  and  $J$ , we have

$$\begin{aligned} &\text{Ann}_{[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}} \prod_i f_i^{s_i} + [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} g \mathbf{a} \\ = &(\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} + (D_Y g \mathbf{a}) \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} \\ = &((\text{Ann}_{[D_Y]} \prod_i f_i^{s_i})^* + D_Y g \mathbf{a}) \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})}. \end{aligned}$$

Hence the condition (2.1) is equivalent to saying that

$$b(\sigma)g \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g \mathbf{a}. \quad (3.1)$$

Since  $t_j \prod_i f_i^{s_i} = f_j \prod_i f_i^{s_i}$ , the condition (2.1) is also equivalent to saying that

$$b(\sigma)g \prod_i f_i^{s_i} \in (V^1 D_Y)g \prod_i f_i^{s_i} = (F_{-1}^{(\mathbf{w}, -\mathbf{w})} D_Y)g \prod_i f_i^{s_i} \quad (3.2)$$

$$\iff b(\sigma) \in \text{in}_{(-\mathbf{w}, \mathbf{w})}(\text{Ann}_{D_Y} g \prod_i f_i^{s_i}) \quad (3.3)$$

$$\iff b(\sigma)g \in \text{in}_{(-\mathbf{w}, \mathbf{w})}((\text{Ann}_{D_Y} \prod_i f_i^{s_i}) \cap D_Y g). \quad (3.4)$$

By the expression (3.2), the generalized Bernstein-Sato polynomial  $b_{\mathbf{a},g}(s)$  coincides with  $b_u(s)$  where  $u = g \otimes 1 \in M_{\mathbf{f}}$ .

By (3.3), the polynomial  $b_{a,g}(-s-r)$  coincides with the  $b$ -function for  $\text{Ann}_{D_Y} g \prod_i f_i^{s_i}$  with the weight vector  $(\mathbf{w}, -\mathbf{w})$  in [18], p.194. In the case  $g = 1$ , one can compute  $b_a(s)$  using loc. cit., p.196, Algorithm 5.1.6 by the next lemma.

**Lemma 3.2.**

$$\text{Ann}_{D_Y} \prod_i f_i^{s_i} = \langle t_i - f_i \mid 1 \leq i \leq r \rangle + \langle \partial_{x_j} + \sum_{i=1}^r \partial_{x_j}(f_i) \partial_{t_i} \mid 1 \leq j \leq n \rangle.$$

*Proof.* One can prove the assertion similarly to the case  $n = 1$ . See [18] Lemma 5.3.3.  $\square$

By this lemma and Lemma 2.6, the homogeneous left ideals

$$(\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g \mathbf{a}, \text{ and } \text{in}_{(-\mathbf{w}, \mathbf{w})}((\text{Ann}_{D_Y} \prod_i f_i^{s_i}) \cap D_Y g),$$

in (3.1) and (3.4) are computable. Therefore we can calculate generalized Bernstein-Sato polynomials if we obtain an algorithm for computing the ideal  $\{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \in J\} \cong J \cap \mathbb{C}[x, \sigma]$  for a given homogeneous ideal  $J \subset D_Y$ . One can compute this in the same way as [18] Algorithm 5.1.6. The algorithm calculates  $J' = J \cap \mathbb{C}[x, \sigma_1, \dots, \sigma_r]$  first where  $\sigma_i = -\partial_{t_i} t_i$ , then computes  $J' \cap \mathbb{C}[x, \sigma]$ . This algorithm requires  $2r$  new variables. We will give an algorithm for computing  $J \cap \mathbb{C}[x, \sigma]$  without computing  $J'$ .

**Lemma 3.3.** *Let  $J$  be a homogeneous left ideal of  $D_Y$ . The following hold:*

- (i)  $\sigma$  is in the center of  $[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$ .
- (ii)  $D_Y[s](J + \langle s - \sigma \rangle) \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})} = J \cap [D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$ .
- (iii)  $\{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \in J\} = D_Y[s](J + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s]$ .

*Proof.* (i) Since the ring  $[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$  is generated by  $\partial_{t_j} t_k$ ,  $1 \leq j, k \leq r$ , over  $D_X$ , and  $\sigma$  commutes with any element of  $D_X$ , it is enough to show that  $\sigma(\partial_{t_j} t_k) = (\partial_{t_j} t_k) \sigma$  for all  $1 \leq j, k \leq r$ .

In the case  $j \neq k$ , we obtain

$$\begin{aligned} (\partial_{t_j} t_k)(\partial_{t_j} t_j) &= \partial_{t_j}^2 t_j t_k, & (\partial_{t_j} t_j)(\partial_{t_j} t_k) &= \partial_{t_j}^2 t_j t_k - \partial_{t_j} t_k, \\ (\partial_{t_j} t_k)(\partial_{t_k} t_k) &= \partial_{t_j} \partial_{t_k} t_k^2 - \partial_{t_j} t_k, & (\partial_{t_k} t_k)(\partial_{t_j} t_k) &= \partial_{t_j} \partial_{t_k} t_k^2, \end{aligned}$$

thus  $(\partial_{t_j} t_k)(\partial_{t_j} t_j + \partial_{t_k} t_k) = (\partial_{t_j} t_j + \partial_{t_k} t_k)(\partial_{t_j} t_k)$ . Hence

$$\begin{aligned} (\partial_{t_j} t_k) \sigma &= -(\partial_{t_j} t_k)(\partial_{t_j} t_j + \partial_{t_k} t_k + \sum_{\ell \neq j, k} \partial_{t_j} t_\ell) \\ &= -(\partial_{t_j} t_j + \partial_{t_k} t_k)(\partial_{t_j} t_k) - (\sum_{\ell \neq j, k} \partial_{t_j} t_\ell)(\partial_{t_j} t_k) = \sigma(\partial_{t_j} t_k). \end{aligned}$$

In the case  $j = k$ , it is obvious that  $(\partial_{t_j} t_j) \sigma = \sigma(\partial_{t_j} t_j)$ . Therefore  $\sigma$  is in the center of  $[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$ .

(ii) The inclusion  $D_Y[s](J + \langle s - \sigma \rangle) \cap [D_Y]_0^{(w, -w)} \supset J \cap [D_Y]_0^{(w, -w)}$  is trivial. We will show the converse inclusion. Let

$$h = \sum P_\ell s^\ell + Q(s)(s - \sigma) \in D_Y[s](J + \langle s - \sigma \rangle) \cap [D_Y]_0^{(w, -w)}$$

where  $P_\ell \in J$  and  $Q(s) \in D_Y[s]$ . Taking the degree zero part, we may assume  $P_\ell \in J \cap [D_Y]_0^{(w, -w)}$  and  $Q[s] \in [D_Y]_0^{(w, -w)}[s]$ . As  $\sum P_\ell s^\ell - \sum P_\ell \sigma^\ell \in [D_Y]_0^{(w, -w)}[s](s - \sigma)$ , there exists  $Q'(s) \in [D_Y]_0^{(w, -w)}[s]$  such that  $h = \sum P_\ell \sigma^\ell + Q'(s)(s - \sigma)$ . Since  $h \in D_Y$ , we have  $Q'(s) = 0$ . Therefore  $h = \sum P_\ell \sigma^\ell = \sum \sigma^\ell P_\ell \in J \cap [D_Y]_0^{(w, -w)}$ .

(iii) Let  $b(\mathbf{x}, \sigma) \in J$ . Since  $b(\mathbf{x}, s) - b(\mathbf{x}, \sigma) \in \langle s - \sigma \rangle$ , we have

$$b(\mathbf{x}, s) = b(\mathbf{x}, \sigma) + (b(\mathbf{x}, s) - b(\mathbf{x}, \sigma)) \in D_Y[s](J + \langle s - \sigma \rangle).$$

Conversely, if  $b(\mathbf{x}, s) \in D_Y[s](J + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s]$ , then

$$b(\mathbf{x}, \sigma) = b(\mathbf{x}, s) - (b(\mathbf{x}, s) - b(\mathbf{x}, \sigma)) \in D_Y[s](J + \langle s - \sigma \rangle).$$

Since  $b(\mathbf{x}, \sigma) \in [D_Y]_0^{(w, -w)}$ , and by (ii), we conclude

$$b(\mathbf{x}, \sigma) \in D_Y[s](J + \langle s - \sigma \rangle) \cap [D_Y]_0^{(w, -w)} = J \cap [D_Y]_0^{(w, -w)} \subset J.$$

□

**Theorem 3.4** (Algorithm for global generalized Bernstein-Sato polynomials 1). *Let*

$$I_{\mathbf{f}} = \langle t_i u_1 - f_i \mid 1 \leq i \leq r \rangle + \langle u_1 \partial_{x_j} + \sum_{i=1}^r \partial_{x_j}(f_i) \partial_{t_i} \mid 1 \leq j \leq d \rangle + \langle u_1 u_2 - 1 \rangle$$

be a left ideal of  $D_Y[u_1, u_2]$ . Then compute the following ideals;

1.  $I_{\mathbf{f},1} = I_{\mathbf{f}} \cap D_Y$ ,
2.  $I_{(\mathbf{f};g),2} = D_Y[s](I_{\mathbf{f},1} + g\mathbf{a} + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s]$ ,
3.  $I_{(\mathbf{f};g),3} = I_{(\mathbf{f};g),2} : g = (I_{(\mathbf{f};g),2} \cap \langle g \rangle) \cdot g^{-1}$ ,
4.  $I_{(\mathbf{f};g),4} = I_{(\mathbf{f};g),3} \cap \mathbb{C}[s]$ .

Then  $b_{\mathbf{a},g}(s)$  is the generator of  $I_{(\mathbf{f};g),4}$ .

*Proof.* By Lemma 3.2 and Lemma 2.6,  $I_{\mathbf{f},1} = (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^*$ . As  $I_{\mathbf{f},1} + D_Y g\mathbf{a}$  is a homogeneous ideal, we have

$$I_{(\mathbf{f};g),2} = \{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g\mathbf{a}\}$$

by Lemma 3.3. Since  $b_{\mathbf{a},g}(s)$  is the minimal generator of the ideal

$$\{b(s) \in \mathbb{C}[s] \mid b(\sigma)g(\mathbf{x}) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g\mathbf{a}\},$$

it follows that  $I_{(\mathbf{f};g),4} = (I_{(\mathbf{f};g),2} : g) \cap \mathbb{C}[s] = \langle b_{\mathbf{a},g}(s) \rangle$ . □

**Theorem 3.5** (Algorithm for global generalized Bernstein-Sato polynomials 2). *Let*

$$\tilde{I}_{\mathbf{f}} = \langle t_i - f_i \mid 1 \leq i \leq r \rangle + \langle \partial_{x_j} + \sum_{i=1}^r \partial_{x_j}(f_i) \partial_{t_i} \mid 1 \leq j \leq d \rangle \subset D_Y,$$

and compute the following ideals;

$$0. \tilde{I}_{(\mathbf{f};g),0} = \tilde{I}_{\mathbf{f}} \cap D_Y g = D_Y[u](u\tilde{I}_{\mathbf{f}} + (1-u)g) \cap D_Y,$$

1.  $\tilde{I}_{(\mathbf{f};g),1} = \text{in}_{(\mathbf{w},-\mathbf{w})}(\tilde{I}_{(\mathbf{f};g),0})$ ,
2.  $\tilde{I}_{(\mathbf{f};g),2} = (\tilde{I}_{(\mathbf{f};g),1} + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s]$ ,
3.  $\tilde{I}_{(\mathbf{f};g),3} = \tilde{I}_{(\mathbf{f};g),2} : g = \tilde{I}_{(\mathbf{f};g),2} \cdot g^{-1}$ ,
4.  $\tilde{I}_{(\mathbf{f};g),4} = \tilde{I}_{(\mathbf{f};g),3} \cap \mathbb{C}[s]$ .

Then  $b_{\mathbf{a},g}(s)$  is the generator of  $\tilde{I}_{(\mathbf{f};g),4}$ .

*Proof.* By Lemma 3.2 and Lemma 2.3,  $\tilde{I}_{(\mathbf{f};g),0} = (\text{Ann}_{D_Y} \prod_i f_i^{s_i}) \cap D_Y g$ . As  $\tilde{I}_{(\mathbf{f};g),1}$  is a homogeneous ideal, we have

$$\tilde{I}_{(\mathbf{f};g),2} = \{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \in \text{in}_{(\mathbf{w},-\mathbf{w})}((\text{Ann}_{D_Y} \prod_i f_i^{s_i}) \cap D_Y g)\}$$

by Lemma 3.3. Since  $\tilde{I}_{(\mathbf{f};g),0} \subset D_Y g$ , we have  $\tilde{I}_{(\mathbf{f};g),1} \subset \text{in}_{(\mathbf{w},-\mathbf{w})}(D_Y g) = D_Y g$ . Hence  $\tilde{I}_{(\mathbf{f};g),2} \subset (D_Y g + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s] = \mathbb{C}[\mathbf{x}, s]g$ , and  $\tilde{I}_{(\mathbf{f};g),2} : g = \tilde{I}_{(\mathbf{f};g),2} \cdot g^{-1}$ . Since  $b_{\mathbf{a},g}(s)$  is the minimal generator of the ideal

$$\{b(s) \in \mathbb{C}[s] \mid b(\sigma)g(\mathbf{x}) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g\mathbf{a}\},$$

it follows that  $\tilde{I}_{(\mathbf{f};g),4} = (\tilde{I}_{(\mathbf{f};g),2} : g) \cap \mathbb{C}[s] = \langle b_{\mathbf{a},g}(s) \rangle$ .  $\square$

**Remark 3.6.** Note that

$$I_{(\mathbf{f};g),3} = \tilde{I}_{(\mathbf{f};g),3} = \{b(\mathbf{x}, s) \in \mathbb{C}[s] \mid b(\mathbf{x}, \sigma)g(\mathbf{x}) \prod_i f_i^{s_i} \in (V^1 D_Y)g(\mathbf{x}) \prod_i f_i^{s_i}\},$$

and in the case  $g = 1$ , it follows that

$$I_{(\mathbf{f};1),2} = \tilde{I}_{(\mathbf{f};1),2} = \{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \prod_i f_i^{s_i} \in (V^1 D_Y) \prod_i f_i^{s_i}\}.$$

We can compute local generalized Bernstein-Sato polynomials similarly to the classical case using primary decompositions (see [11], [12], and [13]).

**Theorem 3.7** (Algorithm for local generalized Bernstein-Sato polynomials). *Let  $I_{(\mathbf{f};g),3} = \tilde{I}_{(\mathbf{f};g),3}$  be the ideals in Theorem 3.4 and Theorem 3.5 with primary decompositions  $I_{(\mathbf{f};g),3} = \cap_{i=1}^{\ell} \mathbf{q}_i$ . Then  $b_{\mathbf{a},g}^{\mathbf{p}}(s)$  is the generator of the ideal*

$$\bigcap_{\mathbf{q}_i \cap \mathbb{C}[\mathbf{x}] \subset \mathbf{p}} \mathbf{q}_i \cap \mathbb{C}[s].$$

*Proof.* We set  $R = \mathbb{C}[\mathbf{x}]$ . By the definition,  $b_{\mathbf{a},g}^{\mathbf{p}}(s)$  is the generator of the ideal

$$\{b(s) \in \mathbb{C}[s] \mid b(\sigma)g(\mathbf{x})h(\mathbf{x}) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y g\mathbf{a} \text{ for } \exists h \notin \mathbf{p}\}.$$

This ideal equals to

$$\begin{aligned} & \{b(s) \in \mathbb{C}[s] \mid b(s)g(\mathbf{x})h(\mathbf{x}) \in I_{(\mathbf{f};g),2} \text{ for } \exists h \notin \mathbf{p}\} \\ &= \{b(s) \in \mathbb{C}[s] \mid b(s)h(\mathbf{x}) \in I_{(\mathbf{f};g),3} \text{ for } \exists h \notin \mathbf{p}\} \\ &= (I_{(\mathbf{f};g),3} \otimes_R R_{\mathbf{p}}) \cap \mathbb{C}[s]. \end{aligned}$$

Since  $I_{(\mathbf{f};g),3} \otimes_R R_{\mathbf{p}} = \bigcap_{\mathbf{q}_i \cap R \subset \mathbf{p}} \mathbf{q}_i \otimes_R R_{\mathbf{p}}$ , and  $\mathbf{q}_i \otimes_R R_{\mathbf{p}} = R_{\mathbf{p}}[s]$  if and only if  $\mathbf{q}_i \cap R \subset \mathbf{p}$ , it follows that  $b_{\mathbf{a},g}^{\mathbf{p}}(s)$  is the generator of the ideal  $(\bigcap_{\mathbf{q}_i \cap R \subset \mathbf{p}} \mathbf{q}_i) \cap \mathbb{C}[s]$ .  $\square$

The algorithm for computing generalized Bernstein-Sato polynomials enables us to solve the membership problem for multiplier ideals, but does not give a system of generators of multiplier ideals. We have to compute  $b_{\mathfrak{a},g}(s)$  for infinitely many  $g$  to obtain a system of generators.

#### 4. ALGORITHMS FOR COMPUTING MULTIPLIER IDEALS

The purpose of this section is to obtain algorithms for computing the system of generators of multiplier ideals. To do this, we modify the definition of Budur–Mustață–Saito’s Bernstein-Sato polynomial.

As in the previous section, let  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$  be a polynomial ring over  $\mathbb{C}$ , and  $\mathfrak{a}$  an ideal with a system of generators  $\mathbf{f} = (f_1, \dots, f_r)$ , and fix the weight vector  $(\mathbf{w}, -\mathbf{w}) \in \mathbb{Z}^{n+r} \times \mathbb{Z}^{n+r}$ ,  $\mathbf{w} = ((0, \dots, 0), (1, \dots, 1)) \in \mathbb{Z}^n \times \mathbb{Z}^r$ . We set  $\delta = 1 \otimes 1 \in M_{\mathbf{f}} = \iota_+ \mathbb{C}[\mathbf{x}]$  and  $\overline{M}_{\mathbf{f}}^{(m)} = (V^0 D_Y) \delta / (V^m D_Y) \delta$ . The induced filtration  $V$  on the  $\overline{M}_{\mathbf{f}}^{(m)}$  is finite by the definition of the  $V$ -filtration (Definition 2.8) as in the case  $m = 1$ . For  $g \in \mathbb{C}[\mathbf{x}]$ , we denote by  $\overline{g \otimes 1}$  the image of  $g \otimes 1 = g\delta$  in  $\overline{M}_{\mathbf{f}}^{(m)}$ .

**Definition 4.1.** We define  $b_{\mathfrak{a},g}^{(m)}(s)$  to be the monic minimal polynomial of the action of  $\sigma$  on  $(V^0 D_Y) \overline{g \otimes 1} \subset \overline{M}_{\mathbf{f}}^{(m)}$ . We define  $b_{\mathfrak{a}}^{(m)} = b_{\mathfrak{a},1}^{(m)}$ .

The existence of  $b_{\mathfrak{a},g}^{(m)}(s)$  follows from the finiteness of the filtration  $V$  on  $\overline{M}_{\mathbf{f}}^{(m)}$ , and the rationality of its roots follows from the rationality of the  $V$ -filtration. Note that  $b_{\mathfrak{a},g}^{(m)}(s) = 1$  if and only if  $g \otimes 1 \in (V^m D_Y) \delta$ .

**Observation 4.2.** Since the ring  $V^0 D_Y$  is generated by  $\mathbf{t} = t_1, \dots, t_r$  over  $[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$ ,  $\sigma$  is in the center of  $[D_Y]_0^{(\mathbf{w}, -\mathbf{w})}$ , and  $t_i \cdot \overline{g \otimes 1} = f_i \cdot \overline{g \otimes 1}$ , it follows that  $b_{\mathfrak{a},g}^{(m)}(s)$  is the monic polynomial  $b(s)$  of minimal degree satisfying  $b(\sigma) \overline{g \otimes 1} = 0$ . This is equivalent to saying

$$b(s)g \prod_i f_i^{s_i} \in V^m D_Y \prod_i f_i^{s_i}.$$

Since  $V^m D_Y$  is generated by all monomials in  $\mathbf{t} = t_1, \dots, t_r$  of degree  $m$  as  $V^0 D_Y$ -module, and  $t_j \prod_i f_i^{s_i} = f_j \prod_i f_i^{s_i}$ , our Bernstein-Sato polynomial  $b_{\mathfrak{a},g}^{(m)}(s)$  is the monic polynomial  $b(s)$  of minimal degree satisfying

$$b(s)g \prod_i f_i^{s_i} = \sum_j P_j h_j \prod_i f_i^{s_i} \quad \text{for } \exists P_j \in [D_Y]_0^{(\mathbf{w}, -\mathbf{w})}, \exists h_j \in \mathfrak{a}^m. \quad (4.1)$$

Hence if  $g, h \in \mathbb{C}[\mathbf{x}]$  are polynomials such that  $g$  divides  $h$ , then  $b_{\mathfrak{a},h}^{(m)}(s)$  is a factor of  $b_{\mathfrak{a},g}^{(m)}(s)$ . In particular,  $b_{\mathfrak{a},g}^{(m)}(s)$  is a factor of  $b_{\mathfrak{a}}^{(m)}(s)$  for all  $g \in \mathbb{C}[\mathbf{x}]$ .

We obtain a description of multiplier ideals in terms of our Bernstein-Sato polynomials similarly to Theorem 2.12.

**Theorem 4.3.** (i) For a given rational number  $c < m + \text{lct}(\mathfrak{a})$ ,

$$\mathcal{J}(\mathfrak{a}^c) = \{g \in \mathbb{C}[\mathbf{x}] \mid c < c' \text{ if } b_{\mathfrak{a},g}^{(m)}(-c') = 0\}.$$

In particular, the log canonical threshold  $\text{lct}(\mathbf{a})$  of  $\mathbf{a} = \langle f_1, \dots, f_r \rangle$  is the minimal root of  $b_1^{(m)}(-s)$ .

(ii) All jumping coefficients of  $\mathbf{a}$  in  $[\text{lct}(\mathbf{a}), \text{lct}(\mathbf{a}) + m)$  are roots of  $b_{\mathbf{a}}^{(m)}(-s)$ .

*Proof.* First note that we have  $\mathcal{J}(\mathbf{a}^c) = V^{c+\varepsilon}\mathbb{C}[\mathbf{x}]$  for  $0 < \varepsilon \ll 1$  by Theorem 2.12. Since  $\text{lct}(\mathbf{a}) = \max\{c \mid \delta \in V^c M_{\mathbf{f}}\}$ , we have  $(V^m D_Y)\delta \subset V^{m+\text{lct}(\mathbf{a})} M_{\mathbf{f}}$ .

If  $g \in \mathbb{C}[\mathbf{x}]$  is a polynomial with  $b_{\mathbf{a},g}^{(m)}(s) = 1$ , then  $g \otimes 1 \in (V^m D_Y)\delta \subset V^{m+\text{lct}(\mathbf{a})} M_{\mathbf{f}}$ . Thus  $g \in \mathcal{J}(\mathbf{a}^c)$  for all  $c < m + \text{lct}(\mathbf{a})$ .

If  $g \notin \mathcal{J}(\mathbf{a}^{m+\text{lct}(\mathbf{a})})$ , then  $g \otimes 1 \notin V^{m+\text{lct}(\mathbf{a})} M_{\mathbf{f}}$ . Hence we obtain

$$\max\{c \mid g \otimes 1 \in V^c M\} = \min\{c \mid b_{\mathbf{a},g}^{(m)}(-c) = 0\}$$

by Remark 2.9. If  $g$  is a polynomial with  $b_{\mathbf{a},g}^{(m)}(-s) = 1$ , then  $g \otimes 1 \in (V^m D_Y)\delta \subset V^{m+\text{lct}(\mathbf{a})} M_{\mathbf{f}}$ . Hence  $g \in V^{m+\text{lct}(\mathbf{a})}\mathbb{C}[\mathbf{x}]$ .

Therefore, for  $c < m + \text{lct}(\mathbf{a})$ , we have

$$V^c \mathbb{C}[\mathbf{x}] = \{g \in \mathbb{C}[\mathbf{x}] \mid c \leq c' \text{ if } b_{\mathbf{a},g}^{(m)}(-c') = 0\}.$$

Thus

$$V^{c+\varepsilon} \mathbb{C}[\mathbf{x}] = \{g \in \mathbb{C}[\mathbf{x}] \mid c < c' \text{ if } b_{\mathbf{a},g}^{(m)}(-c') = 0\}$$

for  $0 < \varepsilon \ll 1$ . This proves the assertion.  $\square$

Our Bernstein-Sato polynomials do not tell us any information about multiplier ideals  $\mathcal{J}(\mathbf{a}^c)$  for  $c \geq m + \text{lct}(\mathbf{a})$ . Our definition, however, has the advantage in that we need just one module  $\overline{M}_{\mathbf{f}}^{(m)}$  to compute Bernstein-Sato polynomials  $b_{\mathbf{a},g}^{(m)}(s)$  for all  $g$ . This fact enables us to compute multiplier ideals.

**Theorem 4.4** (Algorithm for multiplier ideals 1). *Let*

$$I_{\mathbf{f}} = \langle t_i u_1 - f_i \mid 1 \leq i \leq r \rangle + \langle u_1 \partial_{x_j} + \sum_{i=1}^r \partial_{x_j}(f_i) \partial_{t_i} \mid 1 \leq j \leq d \rangle + \langle u_1 u_2 - 1 \rangle$$

be a left ideal of  $D_Y[u_1, u_2]$ . Then compute the following ideals;

1.  $I_{\mathbf{f},1} = I_{\mathbf{f}} \cap D_Y$ ,
2.  $J_{\mathbf{f}}(m) = D_Y[s](I_{\mathbf{f},1} + \mathbf{a}^m + \langle s - \sigma \rangle) \cap \mathbb{C}[\mathbf{x}, s]$ .

Then the following hold:

- (i)  $b_{\mathbf{a},g}^{(m)}(s)$  is the generator of  $(J_{\mathbf{f}}(m) : g) \cap \mathbb{C}[s]$ .
- (ii) Let  $J_{\mathbf{f}}(m) = \cap_{i=1}^{\ell} \mathbf{q}_i$  be a primary decomposition of  $J_{\mathbf{f}}(m)$ . Then, for  $1 \leq i \leq \ell$ , there exists  $c(i)$ , a root of  $b_{\mathbf{a}}^{(m)}(-s)$ , such that the generator of  $\mathbf{q}_i \cap \mathbb{C}[s]$  is some power of  $s + c(i)$ , and

$$\{c(i) \mid 1 \leq i \leq \ell\} = \{c' \mid b_{\mathbf{a}}^{(m)}(-c') = 0\}.$$

- (iii) For  $c < \text{lct}(\mathbf{a}) + m$ ,

$$\mathcal{J}(\mathbf{a}^c) = \bigcap_{i \in \{j \mid c(j) \leq c\}} \mathbf{q}_i \cap \mathbb{C}[\mathbf{x}].$$

*Proof.* (i) As we already see in Observation 4.2 4.1,  $b_{a,g}^{(m)}(s)$  is the monic polynomial  $b(s)$  of minimal degree satisfying

$$b(s)g \prod_i f_i^{s_i} = \sum_j P_j h_j \prod_i f_i^{s_i}$$

for some  $P_j \in [D_Y]_0^{(w,-w)}$  and  $h_j \in \mathfrak{a}^m$ . This is equivalent to

$$\begin{aligned} & (b(\sigma)g - \sum_j P_j h_j) \prod_i f_i^{s_i} = 0 \quad \text{for } \exists P_j \in [D_Y]_0^{(w,-w)}, \exists h_j \in \mathfrak{a}^m \\ \iff & b(\sigma)g - \sum_j P_j h_j \in \text{Ann}_{[D_Y]_0^{(w,-w)}} \prod_i f_i^{s_i} \quad \text{for } \exists P_j \in [D_Y]_0^{(w,-w)}, \exists h_j \in \mathfrak{a}^m \\ \iff & b(\sigma)g \in \text{Ann}_{[D_Y]_0^{(w,-w)}} \prod_i f_i^{s_i} + [D_Y]_0^{(w,-w)} \mathfrak{a}^m \\ & = (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* \cap [D_Y]_0^{(w,-w)} + (D_Y \mathfrak{a}^m) \cap [D_Y]_0^{(w,-w)} \\ & = ((\text{Ann}_{[D_Y]} \prod_i f_i^{s_i})^* + D_Y \mathfrak{a}^m) \cap [D_Y]_0^{(w,-w)} \\ \iff & b(\sigma)g \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y \mathfrak{a}^m. \end{aligned}$$

Hence  $b_{a,g}^{(m)}(s)$  is the generator of the ideal

$$\{b(s) \in \mathbb{C}[s] \mid b(\sigma)g(\mathbf{x}) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y \mathfrak{a}^m\}.$$

On the other hand, we have  $I_{f,1} = (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^*$  by Lemma 3.2 and Lemma 2.6. Since  $(\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y \mathfrak{a}^m$  is a homogeneous ideal, we obtain

$$J_f(m) = \{b(\mathbf{x}, s) \in \mathbb{C}[\mathbf{x}, s] \mid b(\mathbf{x}, \sigma) \in (\text{Ann}_{D_Y} \prod_i f_i^{s_i})^* + D_Y \mathfrak{a}^m\},$$

by Lemma 3.3. Therefore  $b_{a,g}^{(m)}(s)$  is the generator of  $(J_f(m) : g) \cap \mathbb{C}[s]$ .

(ii) Since  $b_a^{(m)}(s)$  is the generator of  $J_f(m) \cap \mathbb{C}[s]$ , and  $\bigcap \mathfrak{q}_i \cap \mathbb{C}[s]$  is a primary decomposition of  $J_f(m) \cap \mathbb{C}[s]$ , we conclude (ii). Note that  $\bigcap \mathfrak{q}_i \cap \mathbb{C}[s]$  is not an irredundant primary decomposition of  $J_f(m) \cap \mathbb{C}[s]$  in general even if  $\bigcap_{i=1}^\ell \mathfrak{q}_i$  is irredundant, and it may happen that  $c(i) = c(j)$  for  $i \neq j$ .

(iii) Let  $J_c$  be the ideal on the right-hand side of the equality. If  $g \in \mathcal{J}(\mathfrak{a}^c)$ , then  $b_{a,g}^{(m)}(s)g \in J_f(m)$  and all the roots of  $b_{a,g}^{(m)}(-s)$  are strictly larger than  $c$ . Suppose that  $\mathfrak{q}_i$  satisfies  $c(i) \leq c$ . Since  $b_{a,g}^{(m)}(s)g \in J_f(m) \subset \mathfrak{q}_i$ , and any power of  $b_{a,g}^{(m)}(s)$  is not in  $\mathfrak{q}_i$ , we have  $g \in \mathfrak{q}_i$ . Hence we conclude  $g \in J_c$ .

For the converse inclusion, let  $g \in J_c$ . If  $c(i) \leq c$ , then  $g \in \mathfrak{q}_i$ , and thus  $\mathfrak{q}_i : g = \mathbb{C}[\mathbf{x}, s]$ . Since  $J_f(m) : g = \bigcap_{i=1}^\ell (\mathfrak{q}_i : g)$  and by (i), all the roots of  $b_{a,g}^{(m)}(-s)$  are also strictly larger than  $c$ . Therefore we have  $g \in \mathcal{J}(\mathfrak{a}^c)$  by Theorem 4.3 (i).  $\square$

**Theorem 4.5** (Algorithm for multiplier ideals 2). *Let the notation be as in Theorem 4.4. Compute  $b_a^{(m)}(s)$ , the generator of  $J_f(m) \cap \mathbb{C}[s]$ , and let  $c_1, \dots, c_\ell$ ,  $c_i < c_{i+1}$ , be*

all the roots of  $b_{\mathbf{a}}^{(m)}(-s)$ . Then, for  $c < \text{lct}(\mathbf{a}) + m$ ,

$$\mathcal{J}(\mathbf{a}^c) = (J_{\mathbf{f}}(m) : \left( \prod_{c < c_i} (s + c_i) \right)^\infty) \cap \mathbb{C}[\mathbf{x}].$$

*Proof.* Let  $g \in \mathcal{J}(\mathbf{a}^c)$ . Then  $b_{\mathbf{a},g}^{(m)}(s)g \in J_{\mathbf{f}}(m)$  and all the roots of  $b_{\mathbf{a},g}^{(m)}(-s)$  are strictly larger than  $c$ . Since  $b_{\mathbf{a},g}^{(m)}(s)$  is a factor of  $b_{\mathbf{a}}^{(m)}(s)$ , it follows that  $g$  is in the ideal on the right hand side. Hence  $\mathcal{J}(\mathbf{a}^c) \subset (J_{\mathbf{f}}(m) : \left( \prod_{c < c_i} (s + c_i) \right)^\infty) \cap \mathbb{C}[\mathbf{x}]$ .

For the converse inclusion, let  $g \in (J_{\mathbf{f}}(m) : \left( \prod_{c < c_i} (s + c_i) \right)^\infty) \cap \mathbb{C}[\mathbf{x}]$ . Then there exists a polynomial  $b(s)$  such that  $b(s)g \in J_{\mathbf{f}}(m)$  and the set of roots of  $b(-s)$  is in  $\{c_i \mid c < c_i\}$ . Since  $b(s) \in (J_{\mathbf{f}}(m) : g) \cap \mathbb{C}[s] = \langle b_{\mathbf{a},g}^{(m)}(s) \rangle$ ,  $b_{\mathbf{a},g}^{(m)}(s)$  is a factor of  $b(s)$ . Hence all the roots of  $b_{\mathbf{a},g}^{(m)}(-s)$  are strictly larger than  $c$ . Therefore we have  $g \in \mathcal{J}(\mathbf{a}^c)$  by Theorem 4.3 (i).  $\square$

**Remark 4.6.** (i) In the case  $m = 1$ ,  $J_{\mathbf{f}}(1)$  coincides with  $I_{(\mathbf{f};1),2}$  in Theorem 3.4 and  $b_{\mathbf{a}}(s) = b_{\mathbf{a}}^{(1)}(s)$ . Since  $I_{(\mathbf{f};1),2} = \tilde{I}_{(\mathbf{f};1),2}$  by Remark 3.6, one can obtain another algorithm in this case.

(ii) Since  $\mathcal{J}(\mathbf{a}^c) = \mathbf{a}\mathcal{J}(\mathbf{a}^{c-1})$  for  $c \geq \lambda(\mathbf{a})$ , it is enough to compute  $J_{\mathbf{f}}(m)$  for  $m$  satisfying  $m \geq \lambda(\mathbf{a}) - \text{lct}(\mathbf{a})$  to obtain all multiplier ideals. In particular, if  $\mathbf{a}$  is a principal ideal, it is enough to compute  $J_{\mathbf{f}}(1)$ .

## 5. EXAMPLES

The computations were made using Kan/sm1 [20] and Risa/Asir [10].

**Example 5.1.** (i) ([19]) Let  $M = (x_{ij})_{ij}$  be the  $n \times n$  general matrix, and  $f = \det M \in \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq r]$ . Then  $b_f(s) = \prod_{i=1}^r (s + i)$ , and  $b_f(\sigma)f^s = f(\partial_{\mathbf{x}})f^{s+1}$ . Here, we use the notation  $h(\partial_{\mathbf{x}})$  to mean  $\sum a_{\mu} \partial_{\mathbf{x}}^{\mu}$  for  $h(\mathbf{x}) = \sum a_{\mu} \mathbf{x}^{\mu}$ .

(ii) Let  $\mathbf{a} = I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix} \subset \mathbb{C}[x_1, \dots, x_6]$  with a system of generators  $(f_1, f_2, f_3) = (x_1x_5 - x_2x_4, x_2x_6 - x_3x_5, x_3x_4 - x_1x_6)$ . Then

$$b_{\mathbf{a}}(s) = (s + 2)(s + 3),$$

and

$$(\sigma + 2)(\sigma + 3)f_1^{s_1}f_2^{s_2}f_3^{s_3} = f_1(\partial_{\mathbf{x}})f_1^{s_1+1}f_2^{s_2}f_3^{s_3} + f_2(\partial_{\mathbf{x}})f_1^{s_1}f_2^{s_2+1}f_3^{s_3} + f_3(\partial_{\mathbf{x}})f_1^{s_1}f_2^{s_2}f_3^{s_3+1}.$$

**Example 5.2.** (i) Let  $f = x^2 + y^3 \in \mathbb{C}[x, y]$ . Then

$$\begin{aligned} b_f(s) &= \left(s + \frac{5}{6}\right)(s + 1)\left(s + \frac{7}{6}\right), \\ b_{f,x}(s) &= (s + 1)\left(s + \frac{11}{6}\right)\left(s + \frac{13}{6}\right), \\ b_{f,y}(s) &= (s + 1)\left(s + \frac{7}{6}\right)\left(s + \frac{11}{6}\right). \end{aligned}$$



(ii) Let  $\mathfrak{a} = \langle x^2, y^3 \rangle \subset \mathbb{C}[x, y]$ . Then

$$\begin{aligned} b_{\mathfrak{a}}(s) &= (s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{4}{3})(s + \frac{3}{2})(s + \frac{5}{3})(s + 2), \\ b_{\mathfrak{a},x}(s) &= (s + \frac{4}{3})(s + \frac{5}{3})(s + \frac{11}{6})(s + 2)(s + \frac{13}{6})(s + \frac{5}{2}), \\ b_{\mathfrak{a},y}(s) &= (s + \frac{7}{6})(s + \frac{3}{2})(s + \frac{5}{3})(s + \frac{11}{6})(s + 2)(s + \frac{7}{3}). \end{aligned}$$

(iii) Let  $f = x_3x^2 + x_4y^3 \in \mathbb{C}[x, y, x_3, x_4]$ . Then

$$\begin{aligned} b_f(s) &= (s + \frac{5}{6})(s + 1)(s + \frac{7}{6})(s + \frac{4}{3})(s + \frac{3}{2})(s + \frac{5}{3})(s + 2), \\ b_{f,x}(s) &= (s + 1)(s + \frac{4}{3})(s + \frac{5}{3})(s + \frac{11}{6})(s + 2)(s + \frac{13}{6})(s + \frac{5}{2}), \\ b_{f,y}(s) &= (s + 1)(s + \frac{7}{6})(s + \frac{3}{2})(s + \frac{5}{3})(s + \frac{11}{6})(s + 2)(s + \frac{7}{3}). \end{aligned}$$

**Example 5.3.** Let  $f = (x + y)^2 - (x - y)^5 \in \mathbb{C}[x, y]$  which is not a non-degenerate polynomial. Then

$$\begin{aligned} b_f(s) &= (s + \frac{7}{10})(s + \frac{9}{10})(s + 1)(s + \frac{11}{10})(s + \frac{13}{10}), \\ b_{f,x}(s) = b_{f,y}(s) &= (s + \frac{9}{10})(s + 1)(s + \frac{11}{10})(s + \frac{13}{10})(s + \frac{17}{10}), \\ b_{f,x+y}(s) &= (s + 1)(s + \frac{17}{10})(s + \frac{19}{10})(s + \frac{21}{10})(s + \frac{23}{10}), \\ b_{f,xy}(s) &= (s + 1)(s + \frac{11}{10})(s + \frac{13}{10})(s + \frac{17}{10})(s + \frac{19}{10}), \end{aligned}$$

and the multiplier ideals are

$$\mathcal{J}(f^c) = \begin{cases} \mathbb{C}[x, y] & 0 \leq c < \frac{7}{10}, \\ \langle x, y \rangle = \langle x + y, x - y \rangle & \frac{7}{10} \leq c < \frac{9}{10}, \\ \langle x + y, xy \rangle = \langle x + y, (x - y)^2 \rangle & \frac{9}{10} \leq c < 1, \end{cases}$$

and  $\mathcal{J}(f^c) = f\mathcal{J}(f^{c-1})$  for  $c \geq 1$ .

**Example 5.4.** Let  $f = xy(x + y)(x + 2y) \in \mathbb{C}[x, y]$ . Then

$$\begin{aligned} b_f(s) &= (s + \frac{1}{2})(s + \frac{3}{4})(s + 1)^2(s + \frac{5}{4})(s + \frac{3}{2}), \\ b_f^{(2)}(s) &= (s + \frac{1}{2})(s + \frac{3}{4})(s + 1)^2(s + \frac{5}{4})(s + \frac{3}{2})(s + \frac{7}{4})(s + 2)^2 \\ &\quad (s + \frac{9}{4})(s + \frac{5}{2}), \\ b_{f,x}(s) = b_{f,y}(s) &= (s + \frac{3}{4})(s + 1)^2(s + \frac{5}{4})(s + \frac{3}{2})(s + \frac{7}{4})(s + 2), \\ b_{f,x^2}(s) = b_{f,y^2}(s) &= (s + 1)^2(s + \frac{5}{4})(s + \frac{3}{2})(s + \frac{7}{4})(s + 2)(s + 3), \end{aligned}$$

and the multiplier ideals are

$$\mathcal{J}(f^c) = \begin{cases} \mathbb{C}[x, y] & 0 \leq c < \frac{1}{2}, \\ \langle x, y \rangle & \frac{1}{2} \leq c < \frac{3}{4}, \\ \langle x, y \rangle^2 & \frac{3}{4} \leq c < 1, \end{cases}$$

and  $\mathcal{J}(f^c) = f\mathcal{J}(f^{c-1})$  for  $c \geq 1$ . Note that  $\frac{5}{4}$  and  $\frac{9}{4}$  are roots of  $b_f^{(2)}(-s)$  which are not jumping coefficients.

**Example 5.5.** Let  $\mathfrak{a} \subset \mathbb{C}[x_1, x_2, x_3]$  be the defining ideal of the space monomial curve  $\text{Spec } \mathbb{C}[T^4, T^5, T^6] \subset \mathbb{C}^3$  with a system of generators  $\mathbf{f} = (x_2^2 - x_1x_3, x_1^3 - x_3^2)$ . Then generalized Bernstein-Sato polynomials are

$$\begin{aligned} b_{\mathfrak{a}}(s) &= (s + \frac{17}{12})(s + \frac{3}{2})(s + \frac{19}{12})(s + \frac{7}{4})(s + \frac{11}{6})(s + \frac{23}{12})(s + 2)(s + \frac{25}{12}) \\ &\quad (s + \frac{13}{6})(s + \frac{9}{4}), \\ b_{\mathfrak{a}, x_1}(s) &= (s + \frac{7}{4})(s + \frac{23}{12})(s + 2)(s + \frac{25}{12})(s + \frac{13}{6})(s + \frac{9}{4})(s + \frac{29}{12})(s + \frac{5}{2}) \\ &\quad (s + \frac{31}{12})(s + \frac{17}{6}), \\ b_{\mathfrak{a}, x_2}(s) &= (s + \frac{11}{6})(s + 2)(s + \frac{13}{6})(s + \frac{9}{4})(s + \frac{29}{12})(s + \frac{5}{2})(s + \frac{31}{12})(s + \frac{11}{4}) \\ &\quad (s + \frac{35}{12})(s + \frac{37}{12}), \\ b_{\mathfrak{a}, x_3}(s) &= (s + \frac{23}{12})(s + 2)(s + \frac{25}{12})(s + \frac{9}{4})(s + \frac{29}{12})(s + \frac{5}{2})(s + \frac{31}{12})(s + \frac{11}{4}) \\ &\quad (s + \frac{17}{6})(s + \frac{19}{6}), \end{aligned}$$

and

$$\begin{aligned} b_{\mathfrak{a}, x_1^2}(s) &= (s + 2)(s + \frac{25}{12})(s + \frac{9}{4})(s + \frac{29}{12})(s + \frac{5}{2})(s + \frac{31}{12})(s + \frac{11}{4})(s + \frac{17}{6}) \\ &\quad (s + \frac{35}{12})(s + \frac{19}{6}), \\ b_{\mathfrak{a}, x_2^2}(s) &= (s + 2)(s + \frac{9}{4})(s + \frac{29}{12})(s + \frac{5}{2})(s + \frac{31}{12})(s + \frac{11}{4})(s + \frac{17}{6})(s + \frac{35}{12}) \\ &\quad (s + \frac{37}{12})(s + \frac{19}{6}). \end{aligned}$$

The ideal  $J_{\mathbf{f}}(1)$  has a primary decomposition  $\cap_{i=1}^{10} \mathfrak{q}_i$  where

$$\begin{aligned} \mathfrak{q}_1 &= \langle 12s + 17, x_1, x_2, x_3 \rangle, \quad \mathfrak{q}_2 = \langle 2s + 3, x_1, x_2, x_3 \rangle, \\ \mathfrak{q}_3 &= \langle 12s + 19, x_1, x_2, x_3 \rangle, \quad \mathfrak{q}_4 = \langle 4s + 7, x_1^2, x_2, x_3 \rangle, \\ \mathfrak{q}_5 &= \langle 6s + 11, x_1, x_2^2, x_3 \rangle, \quad \mathfrak{q}_6 = \langle 12s + 23, x_1^2, x_1x_3, x_2, x_3^2 \rangle, \\ \mathfrak{q}_7 &= \langle s + 2, x_2^2 - x_3x_1, x_1^3 - x_3^2 \rangle, \quad \mathfrak{q}_8 = \langle 12s + 25, x_1^3, x_1x_3, x_2, x_3^2 \rangle, \\ \mathfrak{q}_9 &= \langle 6s + 13, x_1^2, x_2^2, x_3 \rangle, \quad \mathfrak{q}_{10} = \langle 4s + 9, x_1^3, x_1^2x_3, x_1x_2, x_1x_3 - x_2^2, x_2x_3, x_3^2 \rangle, \end{aligned}$$

hence the multiplier ideals are

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{17}{12}, \\ \langle x_1, x_2, x_3 \rangle & \frac{17}{12} \leq c < \frac{7}{4}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{7}{4} \leq c < \frac{11}{6}, \\ \langle x_1^2, x_1 x_2, x_2^2, x_3 \rangle & \frac{11}{6} \leq c < \frac{23}{12}, \\ \langle x_1, x_2, x_3 \rangle^2 & \frac{23}{12} \leq c < 2, \end{cases}$$

and  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq 2$  as  $\mu(\mathfrak{a}) = 2$ . Note that  $\frac{17}{12}, \frac{7}{4}, \frac{11}{6}, \frac{23}{12}, 2$  are all jumping coefficients in  $(0, 2]$ , and  $\frac{3}{2}, \frac{19}{12}, \frac{25}{12}, \frac{13}{6}$ , and  $\frac{9}{4}$  are roots of  $b_{\mathfrak{a}}(-s)$  not coming from the jumping coefficients.

**Example 5.6.** Let  $\mathfrak{a} = \langle x_1^3 - x_2^2, x_2^3 - x_3^2 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$  be the defining ideal of the space monomial curve  $\text{Spec } \mathbb{C}[T^4, T^6, T^9] \subset \mathbb{C}^3$ . Then

$$\begin{aligned} b_{\mathfrak{a}}(s) &= (s + \frac{4}{3})(s + \frac{25}{18})(s + \frac{29}{18})(s + \frac{5}{3})(s + \frac{31}{18})(s + \frac{11}{6})(s + \frac{35}{18})(s + 2) \\ &\quad (s + \frac{37}{18})(s + \frac{13}{6})(s + \frac{41}{18}), \\ b_{\mathfrak{a}, x_1}(s) &= (s + \frac{29}{18})(s + \frac{5}{3})(s + \frac{11}{6})(s + \frac{35}{18})(s + 2)(s + \frac{37}{18})(s + \frac{13}{6})(s + \frac{41}{18}) \\ &\quad (s + \frac{7}{3})(s + \frac{43}{18})(s + \frac{49}{18})(s + \frac{17}{6}), \\ b_{\mathfrak{a}, x_2}(s) &= (s + \frac{31}{18})(s + \frac{35}{18})(s + 2)(s + \frac{37}{18})(s + \frac{13}{6})(s + \frac{41}{18})(s + \frac{7}{3})(s + \frac{43}{18}) \\ &\quad (s + \frac{47}{18})(s + \frac{8}{3})(s + \frac{17}{6})(s + \frac{19}{6}), \end{aligned}$$

and the multiplier ideals are

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{4}{3}, \\ \langle x_1, x_2, x_3 \rangle & \frac{4}{3} \leq c < \frac{29}{18}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{29}{18} \leq c < \frac{31}{18}, \\ \langle x_1^2, x_1 x_2, x_2^2, x_3 \rangle & \frac{31}{18} \leq c < \frac{11}{6}, \\ \langle x_1^3, x_1 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{11}{6} \leq c < \frac{35}{18}, \\ \langle x_1^3, x_1^2 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{35}{18} \leq c < 2, \end{cases}$$

and  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq 2$ .

**Example 5.7.** Let  $\mathfrak{a} = \langle x_1^3 - x_2^2, x_3^2 - x_1^2 x_2 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$  be the defining ideal of the space monomial curve  $\text{Spec } \mathbb{C}[T^4, T^6, T^7] \subset \mathbb{C}^3$ . Then

$$\begin{aligned} b_{\mathfrak{a}}(s) &= (s + \frac{4}{3})(s + \frac{19}{14})(s + \frac{23}{14})(s + \frac{5}{3})(s + \frac{25}{14})(s + \frac{11}{6})(s + \frac{27}{14})(s + 2) \\ &\quad (s + \frac{29}{14})(s + \frac{13}{6})(s + \frac{31}{14}), \end{aligned}$$

and the multiplier ideals are

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{4}{3}, \\ \langle x_1, x_2, x_3 \rangle & \frac{4}{3} \leq c < \frac{23}{14}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{23}{14} \leq c < \frac{25}{14}, \\ \langle x_1^2, x_1 x_2, x_2^2, x_3 \rangle & \frac{25}{14} \leq c < \frac{11}{6}, \\ \langle x_1, x_2, x_3 \rangle^2 & \frac{11}{6} \leq c < \frac{27}{14}, \\ \langle x_1^3, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2 \rangle & \frac{27}{14} \leq c < 2, \end{cases}$$

and  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq 2$ .

**Example 5.8.** Let  $\mathfrak{a} = \langle x_1^4 - x_2^3, x_3^2 - x_1 x_2^2 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$  be the defining ideal of the space monomial curve  $\text{Spec } \mathbb{C}[T^6, T^8, T^{11}] \subset \mathbb{C}^3$ . Then

$$\begin{aligned} b_{\mathfrak{a}}(s) &= \left(s + \frac{9}{8}\right)^2 \left(s + \frac{29}{24}\right) \left(s + \frac{31}{24}\right) \left(s + \frac{11}{8}\right)^2 \left(s + \frac{35}{24}\right) \left(s + \frac{3}{2}\right) \left(s + \frac{37}{24}\right) \left(s + \frac{19}{12}\right) \\ &\quad \left(s + \frac{13}{8}\right)^2 \left(s + \frac{41}{24}\right) \left(s + \frac{43}{24}\right) \left(s + \frac{11}{6}\right) \left(s + \frac{15}{8}\right)^2 \left(s + \frac{23}{12}\right) \left(s + \frac{47}{24}\right) (s + 2) \\ &\quad \left(s + \frac{49}{24}\right) \left(s + \frac{25}{12}\right) \left(s + \frac{13}{6}\right) \left(s + \frac{29}{12}\right), \end{aligned}$$

and the multiplier ideals are

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{9}{8}, \\ \langle x_1, x_2, x_3 \rangle & \frac{9}{8} \leq c < \frac{11}{8}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{11}{8} \leq c < \frac{35}{24}, \\ \langle x_1^2, x_1 x_2, x_2^2, x_3 \rangle & \frac{35}{24} \leq c < \frac{19}{12}, \\ \langle x_1^2, x_1 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{19}{12} \leq c < \frac{13}{8}, \\ \langle x_1^3, x_1 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{13}{8} \leq c < \frac{41}{24}, \\ \langle x_1^3, x_1^2 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{41}{24} \leq c < \frac{43}{24}, \\ \langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, x_2 x_3, x_3^2 \rangle & \frac{43}{24} \leq c < \frac{11}{6}, \\ \langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1^2 x_3, x_2 x_3, x_3^2 \rangle & \frac{11}{6} \leq c < \frac{15}{8}, \\ \langle x_1^4, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1^2 x_3, x_2 x_3, x_3^2 \rangle & \frac{15}{8} \leq c < \frac{23}{12}, \\ \langle x_1^4, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1^2 x_3, x_1 x_2 x_3, x_2^2 x_3, x_3^2 \rangle & \frac{23}{12} \leq c < \frac{47}{24}, \\ \langle x_1^4, x_1^3 x_2, x_1 x_2^2, x_2^3, x_1^2 x_3, x_1 x_2 x_3, x_2^2 x_3, x_3^2 \rangle & \frac{47}{24} \leq c < 2, \end{cases}$$

and  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq 2$ .

**Example 5.9.** Let  $\mathfrak{a} = \langle x_1^2 - x_2 x_3, x_2^2 - x_1 x_3, x_3^2 - x_1 x_2 \rangle = I_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix} \subset \mathbb{C}[x_1, x_2, x_3]$ . Then

$$\begin{aligned} b_{\mathfrak{a}}(s) &= \left(s + \frac{3}{2}\right) (s + 2)^2, \\ b_{\mathfrak{a}}^{(2)}(s) &= \left(s + \frac{3}{2}\right) (s + 2)^2 \left(s + \frac{5}{2}\right) (s + 3)^2, \\ b_{\mathfrak{a}, x_i}(s) &= (s + 2)^2 \left(s + \frac{5}{2}\right), \end{aligned}$$

and the multiplier ideals are

$$J(f^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{3}{2}, \\ \langle x_1, x_2, x_3 \rangle & \frac{3}{2} \leq c < 2, \\ \mathfrak{a} & 2 \leq c < \frac{5}{2}, \\ \langle x_1, x_2, x_3 \rangle \mathfrak{a} & \frac{5}{2} \leq c < 3, \end{cases}$$

and  $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$  for  $c \geq 3$ .

**Example 5.10.** Let  $\mathfrak{a} = \langle x_1^3 - x_2x_3, x_2^2 - x_1x_3, x_3^2 - x_1^2x_2 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$  be the defining ideal of the space monomial curve  $\text{Spec } \mathbb{C}[T^3, T^4, T^5] \subset \mathbb{C}^3$ . Then

$$b_{\mathfrak{a}}(s) = \left(s + \frac{13}{9}\right) \left(s + \frac{3}{2}\right) \left(s + \frac{14}{9}\right) \left(s + \frac{16}{9}\right) \left(s + \frac{17}{9}\right) (s+2)^2 \left(s + \frac{19}{9}\right) \left(s + \frac{20}{9}\right),$$

and

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x_1, x_2, x_3] & 0 \leq c < \frac{13}{9}, \\ \langle x_1, x_2, x_3 \rangle & \frac{13}{9} \leq c < \frac{16}{9}, \\ \langle x_1^2, x_2, x_3 \rangle & \frac{16}{9} \leq c < \frac{17}{9}, \\ \langle x_1^2, x_1x_2, x_2^2, x_3 \rangle & \frac{17}{9} \leq c < 2, \\ \mathfrak{a} & 2 \leq c < \frac{22}{9}. \end{cases}$$

We must compute the ideal  $J_f(2)$  in Theorem 4.4 to obtain multiplier ideals  $\mathcal{J}(\mathfrak{a}^c)$  for  $c \geq \frac{22}{9}$ , and to determine whether  $\frac{22}{9}$  is a jumping coefficient or not. It is, however, a really hard computation.

**Example 5.11.** Let  $f = x^3z^3 + y^3z^2 + y^2 \in \mathbb{C}[x, y, z]$  and  $\mathfrak{a} = \langle f \rangle$ . Then  $b_f(s) = (s + \frac{5}{6})^2(s+1)(s + \frac{7}{6})^2(s + \frac{3}{2})$ , and  $b_{\mathfrak{a},x}(s) = (s + \frac{5}{6})(s+1)(s + \frac{7}{6})^2(s + \frac{3}{2})(s + \frac{11}{6})$ . The ideal  $J_f(1) = I_{(f;1),2}$  has a primary decomposition  $\cap_{i=1}^8 \mathfrak{q}_i$  where

$$\begin{aligned} \mathfrak{q}_1 &= \langle f, s+1 \rangle, \\ \mathfrak{q}_2 &= \langle x, y, 6s+5 \rangle, \quad \mathfrak{q}_3 = \langle x^2, y, 6s+7 \rangle, \\ \mathfrak{q}_4 &= \langle y, z, 6s+5 \rangle, \quad \mathfrak{q}_5 = \langle y, z^2, 6s+7 \rangle, \\ \mathfrak{q}_6 &= \langle x^3, y, z^3, xz, (6s+5)z, (6s+5)x, (6s+5)^2 \rangle, \quad \mathfrak{q}_7 = \langle x^3, y, z^3, 2s+3 \rangle, \\ \mathfrak{q}_8 &= \langle x^3, y, z^3, x^2z^2, (6s+7)z^2, (6s+7)x^2, (6s+7)^2 \rangle. \end{aligned}$$

The ideal  $I_{(f;x),2}$  has primary decomposition  $\cap_{i=1}^7 \mathfrak{q}'_i$  where

$$\begin{aligned} \mathfrak{q}'_1 &= \langle f, s+1 \rangle, \\ \mathfrak{q}'_2 &= \langle x, y, 6s+7 \rangle, \quad \mathfrak{q}'_3 = \langle x^3, y, 6s+11 \rangle, \\ \mathfrak{q}'_4 &= \langle y, z, 6s+5 \rangle, \quad \mathfrak{q}'_5 = \langle y, z^2, 6s+7 \rangle, \\ \mathfrak{q}'_6 &= \langle x^3, y, z^3, xz^2, (6s+7)^2, (6s+7)x, (6s+7)z^2 \rangle, \quad \mathfrak{q}'_7 = \langle x^2, y, z^3, 2s+3 \rangle. \end{aligned}$$

Let  $\mathfrak{p}_1 = \langle x, y \rangle$  and  $\mathfrak{p}_2 = \langle y, z \rangle$  be prime ideals of  $\mathbb{C}[x, y, z]$ , and let  $\mathfrak{m}_0 = \langle x, y, z \rangle$  be the maximal ideal at the origin. Then we obtain local generalized Bernstein-Sato polynomials

$$b_f^{\mathfrak{a}}(s) = s+1, \quad b_f^{\mathfrak{p}_1}(s) = b_f^{\mathfrak{p}_2}(s) = \left(s + \frac{5}{6}\right) (s+1) \left(s + \frac{7}{6}\right), \quad b_f^{\mathfrak{m}_0}(s) = b_f(s),$$

$b_f^{\mathbf{m}}(s) = (s + \frac{5}{6})(s + 1)(s + \frac{7}{6})$  for  $\mathbf{m} \in V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2) \setminus \{\mathbf{m}_0\}$ ,  $b_f^{\mathbf{p}}(s) = s + 1$  for  $\mathbf{p} \in V(\mathfrak{a}) \setminus (V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2))$ , and  $b_f^{\mathbf{p}}(s) = 1$  for  $\mathbf{p} \notin V(\mathfrak{a})$ , and  $b_{\mathfrak{a},x}^{\mathbf{a}}(s) = s + 1$ ,

$$b_{\mathfrak{a},x}^{\mathbf{p}_1}(s) = (s + 1)(s + \frac{7}{6})(s + \frac{11}{6}), \quad b_{\mathfrak{a},x}^{\mathbf{p}_2}(s) = (s + \frac{5}{6})(s + 1)(s + \frac{7}{6}),$$

$b_{\mathfrak{a},x}^{\mathbf{m}_0}(s) = b_{\mathfrak{a},x}(s)$ ,  $b_{\mathfrak{a},x}^{\mathbf{m}}(s) = b_{\mathfrak{a},x}^{\mathbf{p}_i}$  for  $\mathbf{m} \in V(\mathfrak{p}_i) \setminus \{\mathbf{m}_0\}$ ,  $b_{\mathfrak{a},x}^{\mathbf{p}}(s) = s + 1$  for  $\mathbf{p} \in V(\mathfrak{a}) \setminus (V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2))$ , and  $b_{\mathfrak{a},x}^{\mathbf{p}}(s) = 1$  for  $\mathbf{p} \notin V(\mathfrak{a})$ .

**Example 5.12.** Let  $\mathfrak{a}$  be an ideal of  $\mathbb{C}[x, y, z]$  with a system of generators  $\mathbf{f} = (x^3 - y^2z, x^2 + y^2 + z^2 - 1)$ . Then  $b_{\mathfrak{a}}(s) = (s + \frac{11}{6})(s + 2)(s + \frac{13}{6})$ . The ideal  $J_{\mathbf{f}}(1) = I_{(\mathbf{f};1),2}$  is

$$\langle \mathfrak{a}, b_{\mathfrak{a}}(s), (s + 2)y, (s + 2)(6s + 13)x, (s + 2)(z + 1)(z - 1) \rangle,$$

and its primary decomposition is  $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \mathfrak{q}_4 \cap \mathfrak{q}_5$  where

$$\begin{aligned} \mathfrak{q}_1 &= \langle \mathfrak{a}, s + 2 \rangle, \\ \mathfrak{q}_2 &= \langle x, y, z - 1, 6s + 11 \rangle, \quad \mathfrak{q}_3 = \langle x, y, z - 1, 6s + 13 \rangle, \\ \mathfrak{q}_4 &= \langle x, y, z + 1, 6s + 11 \rangle, \quad \mathfrak{q}_5 = \langle x, y, z + 1, 6s + 13 \rangle. \end{aligned}$$

Thus

$$\mathcal{J}(\mathfrak{a}^c) = \begin{cases} \mathbb{C}[x, y, z] & 0 \leq c < \frac{11}{6}, \\ \langle x, y, z^2 - 1 \rangle & \frac{11}{6} \leq c < 2. \end{cases}$$

Let  $\mathbf{m}_1 = \langle x, y, z - 1 \rangle$  and  $\mathbf{m}_2 = \langle x, y, z + 1 \rangle$  be maximal ideals of  $\mathbb{C}[x, y, z]$ . Then we obtain local generalized Bernstein-Sato polynomials

$$b_{\mathfrak{a}}^{\mathbf{a}}(s) = s + 2, \quad b_{\mathfrak{a}}^{\mathbf{m}_1}(s) = b_{\mathfrak{a}}^{\mathbf{m}_2}(s) = (s + \frac{11}{6})(s + 2)(s + \frac{13}{6}),$$

$b_{\mathfrak{a}}^{\mathbf{m}}(s) = s + 2$  for  $\mathbf{m} \in V(\mathfrak{a}) \setminus \{\mathbf{m}_1, \mathbf{m}_2\}$ , and  $b_{\mathfrak{a}}^{\mathbf{p}}(s) = 1$  for  $\mathbf{p} \notin V(\mathfrak{a})$ .

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